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Mathematical

Reviews

Edited by

O. Neugebauer   J. D. Tamarkin   O. Veblen

W. Feller, *Executive Editor*

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Mathematical Reviews

ALGEBRA

Poivert, Jules. Les triangles arithmétiques. Rev. Trimest. Canad. 30, 129–136 (1944). [MF 10681]

Mirimanoff, D. Expression du produit de deux indéterminées en fonction de la somme. Comment. Math. Helv. 15, 45–58 (1943).

Let $f_r = \sum x_1 \cdots x_r$, $(r=1, \cdots, n)$ be the elementary symmetric functions on the n quantities $x_1, \cdots, x_n$, and let $p = x_1 + x_2$, $q = x_1 x_2$. Each of p, q is known to be a rational function of the other, the numerator and denominator having as coefficients polynomials in $f_1, \cdots, f_n$ with rational integral coefficients. In two earlier works [Comment. Math. Helv. 14, 1–22, 310–313 (1942); these Rev. 3, 259], p was expressed in terms of q and in terms of $q + cp$ ($c =$ constant) in especially simple form. The present paper gives a corresponding treatment for q as function of p.
I. M. Sheffer (State College, Pa.).

*Artin, Emil. Galois Theory. Second edition. Edited and supplemented with a section on applications by Arthur N. Milgram. Notre Dame Mathematical Lectures, no. 2. University of Notre Dame, Notre Dame, Ind., 1944. 82 pp. $1.25.

This edition differs from the first [these Rev. 4, 66] chiefly in the following additions: (1) an elegant treatment of determinants, starting from three postulates, and containing as an essential step the demonstration of the theorem on the determinant of a product; (2) a proof of the decomposition theorem for Abelian groups with a finite number of generators, with applications to the theory of finite fields; (3) various added examples to the fundamental theorems of the Galois theory. *S. MacLane.*

Lipka, Stephan. Über die Lage der Wurzeln von algebraischen Gleichungen. Monatsh. Math. Phys. 50, 125–127 (1941). [MF 10476]

This paper establishes the following refinement of the well-known Cauchy theorem on the maximum modulus of the zeros of a given polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \qquad a_n > 0.$$

Let ξ be the positive root of the equation

$$z^n - |a_1| z^{n-1} - \cdots - |a_{n-1}| z - |a_n| = 0$$

and ξ_1 that of the equation

$$z^{n-1} - |a_1| z^{n-2} - \cdots - |a_{n-2}| z - |a_{n-1}| = 0.$$

Clearly, $\xi_1 < \xi$. Let Z be the gear-shaped region obtained from the circle $|z| \leq \xi$ by deleting the n regions $\xi_1 \leq |z| \leq \xi$, $(4k-1)\pi/2n \leq \arg z \leq (4k+1)\pi/2n$ ($k=0, 1, 2, \cdots, n-1$). Then all the zeros of $f(z)$ lie in Z. The proof of this theorem is based upon a lemma [St. Lipka, Acta Lit. Sci. Szeged 5, 69–77 (1931), in particular, p. 72] that two functions $f(z)$ and $g(z)$, regular in a simply-connected region B, have the same number of zeros in B if $\Re(f/g)$ has a constant sign on the boundary of B. *M. Marden* (Milwaukee, Wis.).

identical with the paper in p. 30.

Eckmann, Beno. Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen. Comment. Math. Helv. 15, 358–366 (1943).

The problem is to determine the pairs of positive integers p and n such that there exist n bilinear forms $z_1, \cdots, z_n$ in $x_1, \cdots, x_p$ and $y_1, \cdots, y_n$ such that

$$(x_1^2 + \cdots + x_p^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_n^2$$

is an identity in the x's and y's. The problem was solved by A. Hurwitz and J. Radon. Let $n = u \cdot 2^{4\alpha+\beta}$, where u is odd and $0 \leq \beta < 4$; then the required bilinear forms exist when and only when $p \leq 8\alpha + 2^\beta$. Following Hurwitz and Radon in replacing the problem by an equivalent problem in orthogonal matrices, the author uses the theory of matrix representations (developed for quantum mechanics) to solve the problem. *R. P. Agnew* (Ithaca, N. Y.).

Littlewood, D. E. On the number of terms in a simple algebraic form. Proc. Cambridge Philos. Soc. 38, 394–396 (1942). [MF 7807]

In an earlier paper [same Proc. 38, 129–143 (1942); these Rev. 3, 305] Hodge gave a formula for the number of independent terms in a k-complex, but while conjecturing its general validity he proved it only for special cases. The author points out that the general formula is equivalent to a well-known formula for the degrees of the representations of the full linear group. Incidentally, this formula has already been proved by I. Schur in his thesis [Berlin, 1901]. *R. Brauer* (Toronto, Ont.).

Petiau, Gérard. Sur les matrices de spin. C. R. Acad. Sci. Paris 213, 863–866 (1941). [MF 9660]

The author discusses some algebraic properties of a system of matrices entering in the theory of a particle of arbitrary spin. *A. H. Taub* (Princeton, N. J.).

Kemmer, N. The algebra of meson matrices. Proc. Cambridge Philos. Soc. 39, 189–196 (1943). [MF 9200]

The complete lists of irreducible representations of the algebra generated by s matrices β_p satisfying the equation arising in meson theory, namely,

$$\beta_\lambda \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda = \beta_\lambda \delta_{\mu\nu} + \beta_\nu \delta_{\mu\lambda},$$

are deduced. The representations are constructed by use of antisymmetric tensors in an s dimensional space. *A. H. Taub* (Princeton, N. J.).

Clippinger, R. F. Matrix products of matrix powers. Bull. Amer. Math. Soc. 50, 368–372 (1944). [MF 10600]

Let $\mathfrak{L}$ denote the set of all matrices

$$A(t) = \sum_1^m \rho_i(t) \cdot A_i,$$

where A_i are $n \times n$ matrices with complex elements and $\rho_i(t)$ are arbitrary nonnegative summable functions of the

real variable t on the interval $a \leq t \leq b$. Denote by $\mathfrak{J}, \mathfrak{S}$ or $\mathfrak{X}$ the respective subsets of $\mathfrak{L}$ obtained when the $\rho_i(t)$ are polynomials, step functions or step functions which are all 0 except one. Denote by λ, ι, σ or ξ the set of matrices $Y(t)$ which are particular values of solutions of the differential equation $dY(t)/dt = Y(t) \cdot A(t)$, $Y(a) = E =$ unit matrix, in the respective cases when $A(t)$ is in $\mathfrak{L}, \mathfrak{J}, \mathfrak{S}$ or $\mathfrak{X}$. If $|\exp A_i - E| < 1$ for $i = 1, \cdots, m$, denote by μ the set of matrix products of matrix powers

$$\prod_{j=1}^{J} \prod_{i=1}^{m} (\exp A_i)^{\alpha_{ij}},$$

where the α_{ij} are arbitrary nonnegative numbers. Considering a set β of matrices as a point set in $2n^3$-Euclidean space, denote its closure by $\bar{\beta}$. The author proves that the sets $\bar{\lambda}, \bar{\iota}, \bar{\sigma}, \bar{\xi}$ and (if it exists) $\bar{\mu}$ are identical.

C. C. MacDuffee (Madison, Wis.).

Allen, H. S. Maximum matrix rings. J. London Math. Soc. **18**, 142–147 (1943). [MF 10365]

A set α of sequences of complex numbers is a space if it is closed under addition and under multiplication by complex numbers. The space of all sequences x in which $x_k \neq 0$ for only a finite number of values of k is denoted by φ. The space of all sequences y such that $\sum |x_k y_k|$ is convergent for every $x \epsilon \alpha$ is the dual space α^*. The matrix A applies absolutely to x if the series $\sum |a_{nk} x_k|$ converges for every n. Denote by $\sum(\alpha)$ the space of all matrices which apply absolutely to every sequence in α and transform it into a sequence in α. If $\varphi \subseteq \beta \subseteq \alpha^*$, denote by $\sum_\beta(\alpha)$ the space (ring) of all matrices $A \epsilon \sum(\alpha)$ such that $A' \epsilon \sum(\beta)$ and $u'(Ax) = (u'A)x$ for $x \epsilon \alpha$ and $u \epsilon \beta$. Denote by $S(\alpha, \beta)$ the space of all sequences x such that, for A, B in $\sum_\beta(\alpha)$, (i) A applies absolutely to x, (ii) B applies absolutely to Ax, (iii) $B(Ax) = (BA)x$. The author now proves that the ring $\sum_\beta(\alpha)$ is maximal if and only if $\alpha \supseteq \varphi$, $S(\alpha, \beta) = \alpha$ and $S(\beta, \alpha) = \beta$. He proves further that every maximal ring is a space of the type $\sum_\beta(\alpha)$. *C. C. MacDuffee.*

Abstract Algebra

Stöhr, Alfred. Über zweifach geordnete Mengen und Zerlegungen in Rechtecke. I. J. Reine Angew. Math. **184**, 138–157 (1942). [MF 9038]

A rectangle may be subdivided into a finite set of non-overlapping rectangles with edges parallel to those of the original rectangle. The rectangles of the subdivision may be ordered with respect to the two relations "to the left of" and "below." The author states a number of propositions which may be taken as postulates for a theory of doubly ordered systems covering the above intuitive situation. The postulates are examined for dependence and some consequences are deduced. *L. W. Cohen.*

Schutzenberger, Marcel-Paul. Sur la théorie des structures de Dedekind. C. R. Acad. Sci. Paris **216**, 717–718 (1943). [MF 9994]

This paper contains various statements on the properties of complemented Dedekind structures. Several of the statements appear to be known and others are deducible in a simple manner from the representation of projective geometries by such systems. *O. Ore.*

Bernstein, B. A. Postulate-sets for Boolean rings. Trans. Amer. Math. Soc. **55**, 393–400 (1944). [MF 10499]

The author discusses 12 postulate sets for Boolean rings, starting with one of Stone and two of Stabler. It is shown that the sets are all equivalent, that one postulate can be omitted from Stone's set, and that each of the 12 sets is independent, even after the addition of a unit-element postulate. *L. H. Loomis* (Cambridge, Mass.).

Walfisz, Arnold. Über primäre Ideale. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi] **2**, 383–388 (1941). (German. Russian and Georgian summaries) [MF 10291]

The discussion concerns a certain ideal q which B. L. van der Waerden [Moderne Algebra, vol. 2, Springer, Berlin, 1931, pp. 31–35 or vol. 2, 2nd ed., 1940, pp. 26–30] cites as an example of a primary ideal which is not strongly primary. The author points out that, whereas van der Waerden establishes that q is primary, the proof that q is not strongly primary is inadequate. Moreover, the author proves that q is strongly primary. A correct example of a primary ideal which is not strongly primary has been given by R. Hölzer [Math. Ann. **96**, 719–735 (1927)]. A point which should be mentioned is that W. Krull in his "Idealtheorie" [Ergebnisse der Mathematik, vol. 4, Springer, Berlin, 1935, pp. 30–33] claims to refer to E. Noether [Math. Ann. **96**, 26–61 (1926)] for the concept of a strongly primary ideal and yet states a nonequivalent definition. The definitions are not equivalent because van der Waerden proves in effect that q is not strongly primary in the Krull sense. However, a strongly primary ideal in the Krull sense is also strongly primary in the Noether sense. *I. Niven.*

Chevalley, Claude. On the notion of the ring of quotients of a prime ideal. Bull. Amer. Math. Soc. **50**, 93–97 (1944). [MF 9892]

Let $\mathfrak{o}$ be a commutative ring with unit. Let $\mathfrak{s}$ be the set of zero divisors in $\mathfrak{o}$ different from zero. Let $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. A ring of quotients $\mathfrak{o}'$ can be defined whose elements are the fractions whose denominators do not belong to $\mathfrak{s}$ or $\mathfrak{u}$. If $\mathfrak{s} = 0$ then $\mathfrak{o}'$ is denoted by $\mathfrak{o}_\mathfrak{u}$. It is a local ring in the sense of Krull, that is, the nonunits in $\mathfrak{o}'$ form an ideal. This is not true in the case when $\mathfrak{s} \neq 0$. The problem of defining a "ring of quotients" in the general case which retains this property and which coincides with $\mathfrak{o}_\mathfrak{u}$ in the case $\mathfrak{s} = 0$ is solved in this paper in the case of a Noetherian ring (that is, a ring in which every ideal has a finite set of generators). This construction has numerous applications in algebraic geometry.

The author considers any multiplicatively closed set S not containing 0. Let $\mathfrak{6}$ be the intersection of all primary ideals in $\mathfrak{o}$ which do not meet S and let S^* be the set of residue classes of S modulo $\mathfrak{6}$. Since no element of S^* is a zero divisor in the residue class ring $\mathfrak{o}/\mathfrak{6}$ we may construct the ring $(\mathfrak{o}/\mathfrak{6})_{S^*}$ using the above definitions. This ring is shown to have the properties announced above in the case when S is the complement of a prime ideal. *O. Todd-Taussky.*

Johnson, R. E. On the equation $\chi\alpha = \gamma\chi + \beta$ over an algebraic division ring. Bull. Amer. Math. Soc. **50**, 202–207 (1944). [MF 10198]

The author solves the equation $\chi\alpha = \gamma\chi + \beta$ in a division ring R which is algebraic over its separable center F. If α and γ are not transforms of each other, there is a unique solution, which is given explicitly. If α and γ are transforms

of each other, solution of the given equation depends upon the solution of $\chi \alpha = \alpha \chi + \beta$. If α is in R (but not in F) with minimum polynomial $a(\lambda) \cdot (\lambda - \alpha)$ and $\beta \neq 0$ in R, $\chi \alpha = \alpha \chi + \beta$ has a solution χ in R if and only if $a^r(\beta \alpha \beta^{-1}) = 0$, where the superscript r means substitution on the right into $a(\lambda)$.

C. C. *MacDuffee* (Madison, Wis.).

Tschebotaröw, N. A theorem of the theory of semi-simple Lie groups. Rec. Math. [Mat. Sbornik] N.S. 11(53), 239–244 (1942). (English. Russian summary) [MF 8369]

Let $\mathfrak{g}$ be a semi-simple Lie algebra and let α, β, $\cdots$ be the roots of $\mathfrak{g}$ with respect to a maximal Abelian subalgebra $\mathfrak{a}$. A set Σ_0 of roots is said to be conditionally additive if the sum of two roots in Σ_0 belongs to Σ_0 whenever it is a root. The roots being represented as usual by vectors, a set of roots is said to be one sided if there exists a hyperplane which leaves all roots in the set on one side. It is proved that a conditionally additive set which does not contain any pair of opposite roots is one sided. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be separable if all elements of $\mathfrak{h}$ are linear combinations of elements $\mathfrak{h} \cap \mathfrak{a}$ and of root-operators contained in $\mathfrak{h}$. As an application of the previous theorem, it is proved that a separable $\mathfrak{h}$ is nilpotent if and only if the following conditions are satisfied: (a) the set of roots whose root operators are in $\mathfrak{h}$ does not contain any pair of opposite roots; (b) the elements of $\mathfrak{h} \cap \mathfrak{a}$ are in the center of $\mathfrak{h}$. Condition (a) alone is necessary and sufficient in order for $\mathfrak{h}$ to be solvable.

C. *Chevalley* (Princeton, N. J.).

Albert, A. A. Algebras derived by non-associative matrix multiplication. Amer. J. Math. 66, 30–40 (1944). [MF 9937]

Let $\mathfrak{C}$ be an (associative or nonassociative) algebra over a field $\mathfrak{F}$ and assume that $\mathfrak{C}$ possesses an involution J, that is, an anti-automorphism of period two. Using the elements of $\mathfrak{C}$, three new algebras $\mathfrak{C}_\rho = \mathfrak{C}_\rho(J)$, $\mathfrak{C}_x = \mathfrak{C}_x(J)$, $\mathfrak{C}_{x\rho} = \mathfrak{C}_{x\rho}(J)$ are defined, respectively, by the product formulas

$$x \cdot y = x(yJ), \quad (x, y) = (xJ)y, \quad [x, y] = (xJ)(yJ),$$

the product on the right hand side of each equation being a product in $\mathfrak{C}$. For example, take for $\mathfrak{C}$ the complete matric algebra of degree n, and let J denote the transition to the transposed matrix. Then $\mathfrak{C}_\rho$ is the algebra consisting of all n-rowed matrices in which the product of two matrices is defined by row-by-row multiplication instead of the ordinary row-by-column multiplication. Similarly, $\mathfrak{C}_x$ corresponds to column-by-column multiplication, and $\mathfrak{C}_{x\rho}$ to column-by-row multiplication. Of course, for $n > 1$, the algebras $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$ are nonassociative. For arbitrary $\mathfrak{C}$ and J, the algebras $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$ are isotopic to $\mathfrak{C}$ [cf. A. A. Albert, Ann. of Math. (2) 43, 685–707; these Rev. 4, 186]. Assume that $\mathfrak{C}$ contains a unity element. It is shown that, when $\mathfrak{C}$ is simple, so are $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$. The converse is not true, but it is proved that $\mathfrak{C}$ is semisimple if and only if the algebras $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$ are all semisimple. If $\mathfrak{C}$ is not semisimple and $\mathfrak{R}$ is its radical [cf. A. A. Albert, Bull. Amer. Math. Soc. 48, 891–897 (1942); these Rev. 4, 130], then $\mathfrak{R}J = \mathfrak{R}$, and the algebras $\mathfrak{R}_\rho$, $\mathfrak{R}_x$, $\mathfrak{R}_{x\rho}$ are the respective radicals of $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$, and the difference algebras $\mathfrak{C} - \mathfrak{R}$, $\mathfrak{C}_\rho - \mathfrak{R}_\rho$, $\mathfrak{C}_x - \mathfrak{R}_x$, $\mathfrak{C}_{x\rho} - \mathfrak{R}_{x\rho}$ are all semisimple. If $\mathfrak{C}$ is associative, the assumption concerning the existence of a unity quantity is unnecessary. By a row algebra (relative to $\mathfrak{C}$ and J), the author means a subalgebra of $\mathfrak{C}_\rho(J)$. As shown by suitable examples, neither the algebra $\mathfrak{A}$ nor its radical need be mapped on itself by J. If $\mathfrak{A}J = \mathfrak{A}$, then $\mathfrak{A} = \mathfrak{D}$, for a suitable subalgebra $\mathfrak{D}$ of $\mathfrak{C}$. It is shown that $\mathfrak{B} = \mathfrak{A} \cdot \mathfrak{A}$ (the subalgebra of $\mathfrak{A}$ generated by all $x \cdot y$ with x, y in $\mathfrak{A}$) has the property $\mathfrak{B}J = \mathfrak{B}$. In particular, if $\mathfrak{A} \cdot \mathfrak{A} = \mathfrak{A}$, then $\mathfrak{A} = \mathfrak{D}$, for a suitable subalgebra $\mathfrak{D}$ of $\mathfrak{C}$. A structure theorem is given for the case that $\mathfrak{A} \cdot \mathfrak{A}$ is not equal to $\mathfrak{A}$. If $\mathfrak{F}$ is a real field and $\mathfrak{C}$ the complete matric algebra, a complete construction of all row algebras $\mathfrak{A}$ with $\mathfrak{A} \cdot \mathfrak{A} = \mathfrak{A}$ can be given. R. *Brauer*.

Kolchin, E. R. Extensions of differential fields. II. Ann. of Math. (2) 45, 358–361 (1944). [MF 10272]

[The first part appeared in the same Ann. (2) 43, 724–729 (1942); these Rev. 4, 72.] It is proved that, if $\mathfrak{F}$ is a differential field of characteristic zero, y an unknown and F_1 a differential field such that $\mathfrak{F} \subset \mathfrak{F}_1 \subseteq \mathfrak{F}(y)$, there exists an element ω in $\mathfrak{F}_1$ such that $\mathfrak{F}_1 = \mathfrak{F}(\omega)$. Thus the analogue, for differential fields, of Lüroth's theorem, which previously existed only for the case of meromorphic coefficients, is established in the abstract and, indeed, with a simplicity which was lacking in the restricted treatment. Examples are given to show that the theorem does not carry over to partial differential fields. J. F. *Ritt*.

THEORY OF GROUPS

Miller, G. A. Relative number of non-invariant operators in a group. Proc. Nat. Acad. Sci. U. S. A. 30, 25–28 (1944). [MF 10112]

A non-Abelian group of order g has at least $3g/4$ non-invariant operators. After discussing those groups which have exactly this number, the author proves that every group in which the total number of noninvariant operators is a prime number contains no invariant operator other than the identity. G. de B. *Robinson* (Ottawa, Ont.).

Wade, T. L. and Bruck, R. H. Types of symmetries. Amer. Math. Monthly 51, 123–129 (1944). [MF 10162]

Discussion of the types of symmetries of tensors, in particular, of third and fourth orders, by means of Young's tableaux. R. *Brauer* (Toronto, Ont.).

Certaine, Jeremiah. The ternary operation $(abc) = ab^{-1}c$ **of a group.** Bull. Amer. Math. Soc. 49, 869–877 (1943). [MF 9681]

Let G be a set of elements on which there is defined a ternary operation (abc) [cf. Baer, J. Reine Angew. Math. 160, 199–207 (1929)] satisfying the following postulates: (i) $((abc)de) = (ab(cde))$, (ii) $(abb) = a$, (iii) $(bba) = a$. These postulates are shown to be independent and consistent. It is proved that any element u of G may be chosen as the identity of a group G_u defined by $ab = (aub)$ and that $(abc) = ab^{-1}c$. Further, the groups G_u are isomorphic. Equivalent and weaker postulates are also considered. A geometrical interpretation of the ternary operation leads to an abstract definition of a set of free vectors. It is proved that any group may be converted into a group of free vectors and conversely. D. E. *Rutherford* (St. Andrews).

Baer, Reinhold. The higher commutator subgroups of a group. Bull. Amer. Math. Soc. 50, 143–160 (1944). [MF 10123]

In this address before the American Mathematical Society the author discusses interrelations between various families of subgroups. Besides unsolved problems and conjectures, several new results are stated, their proofs appear-

ing in appendices. The topics of the address are as follows. (1) Classification of subgroups of G as normal, characteristic, strictly characteristic (invariant under endomorphisms whose range is G) and fully invariant (invariant under all endomorphisms). An example shows the latter two to be distinct. (2) The commutator subgroups: the theorem that a finite nilpotent group is a product of p-groups is generalized to infinite groups. (3) The structure of F/F^a for F a free group: conjectures of Burnside that F/F^a is finite and solvable are studied. (4) Hopf's result that $(F, F)/(F, N)$ is an invariant of $G = F/N$ is generalized to higher commutator subgroups. A fairly extensive bibliography of recent literature is appended.   *I. Kaplansky.*

Baer, Reinhold. Groups without proper isomorphic quotient groups. Bull. Amer. Math. Soc. **50**, 267–278 (1944). [MF 10210]

The author obtains certain sufficient conditions for a group G to be a Q-group, that is, a group having no isomorphic quotient groups. The possible complications are indicated by examples showing that neither subgroups nor quotient groups of Q-groups need to be Q-groups, and that the possession of a finite number of generators is neither necessary nor sufficient. Use is made of the following refinement of the notion of characteristic subgroup: a subgroup of G is strictly (completely) characteristic if it admits (is invariant under) every endomorphism whose range is G. Then G is a Q-group if there exists between 1 and G a well-ordered descending (ascending) chain of strictly (completely) characteristic subgroups with the successive quotient groups Q-groups. Application of these criteria to the central series yields special results, including, in particular, that the free group on a finite number of generators is a Q-group. A similar analysis is made of groups having no isomorphic subgroups (S-groups). The paper concludes with a study of the partial duality between ascending and descending series afforded by the centralizer of a subgroup. It is thus shown that, if the ascending central series terminates in G, then the derived series terminates in 1 (a generalization of the theorem that nilpotent groups are soluble). Misprint: the definition of soluble group at the foot of page 276 should read $G^{(s)} = 1$.   *I. Kaplansky.*

Brenner, Joel. The linear homogeneous group. II. Ann. of Math. (2) **45**, 100–109 (1944). [MF 9836]

The linear homogeneous group $\mathfrak{G}_{p,n,r}$ may be represented by $n \times n$ matrices M whose elements are residue classes mod p^r and whose determinants are not equal to 0 (mod p). In an earlier paper [Ann. of Math. (2) **39**, 472–493 (1938)] the author found the normal subgroups of $\mathfrak{G}_{p,n,r}$. In this paper the known results are recast in a different group-theoretic form and further results about normal and characteristic subgroups are obtained. The underlying vector space is an additive Abelian group $\mathfrak{A}$ of type $(p^r, p^r, \cdots, p^r)$ and the group $\mathfrak{G}_{p,n,r}$ is thought of as the group of automorphisms of $\mathfrak{A}$ and may be represented by the matrices M described above. If E is the unit matrix and $a \not\equiv 0 \pmod p$, then the matrices of the form $aE + p^s M$ ($0 \leq s \leq r$) form a normal subgroup $\mathfrak{N}_s{}^*$ and the matrices of the form $E + p^s M$ having unit determinant (mod p^r) form a normal subgroup $\mathfrak{N}_s$. In particular, $\mathfrak{N}_0{}^* = \mathfrak{G}$ and $\mathfrak{N}_r = E$. Every normal subgroup of $\mathfrak{G}$ lies between $\mathfrak{N}_s$ and $\mathfrak{N}_s{}^*$ for some s. The author states 22 theorems concerned with the relationships between the various normal subgroups and exhibits by diagrams the relationships of inclusion among

the normal subgroups for the cases $n = 2$, $p^r = 27$; $n = 5$, $p^r = 9$; $n > 2$, $p^r = 8$; $n = p = 2$, $r \geq 3$. This latter is the most difficult case to describe. Among the typical theorems are the following [Theorems 13 and 14]. For $p > 2$, $n > 1$, the normal subgroups $\mathfrak{N}_s{}^*$ and $\mathfrak{N}_s$ are all characteristic.

                    J. S. Frame (East Lansing, Mich.).

Krull, Wolfgang. Über separable, insbesondere kompakte separable Gruppen. J. Reine Angew. Math. **184**, 19–48 (1942). [MF 9031]

The present paper is an extension of the theory of primary separable closed Abelian groups defined in an earlier paper [same J. **182**, 235–241 (1940); these Rev. **2**, 308]. These groups have the ring of p-adic integers as operator domain and their theory essentially includes the theory of zero-dimensional compact groups. A part of the paper is concerned with foundations and, in particular, with an investigation of infinite direct sums. The group B is said to be a direct sum $B_1 + B_2 + \cdots$ if each formal sum $\beta_1 + \beta_2 + \cdots$, β_i in B_i, converges and B, the totality of these sums, is the smallest closed subgroup containing all the B_i. The smallest closed subgroup $\sum' B_i$ containing all the B_i is a direct sum of the B_i if and only if $\prod_i(\sum_{k \neq i} B_k) = 0$ for all i. The author proves that any group that is "correctly knotted" in the sense defined in the earlier paper is a direct sum of cyclic groups. A separable closed Abelian group is compact if and only if it has no elements of infinite height. For these groups a complete set of invariants may be given by using the duality theorems and Ulm's theory of denumerable torsion groups. These results are applied to the theory of infinite Abelian extensions of a field. It is proved that, if A is any compact and separable zero dimensional Abelian group, then there exist two absolute algebraic fields $\mathfrak{R}$ and $\mathfrak{R}$ such that $\mathfrak{R}$ is Abelian over $\mathfrak{R}$ with Galois group isomorphic to A.   *N. Jacobson* (Baltimore, Md.).

Morosov, V. On a theorem of E. Cartan. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 335–339 (1943). (English. Russian summary) [MF 10227]

The theorem of Cartan referred to in the title is that in an irreducible N-dimensional representation of an infinitesimal semi-simple Lie group there exist N linearly independent weight vectors. Weyl gave an elegant geometrical proof of this theorem and the author gives a matrix-algebraic proof of it. The author also obtains the following two consequences of this theorem. (1) If G is an irreducible infinitesimal linear Lie group of n-rowed matrices then, if $n \leq 7$, G contains a matrix whose characteristic numbers are different. (2) If g is a semi-simple Lie group, g_1 its semi-simple subgroup and h_1 the maximal regular Abelian subgroup of g_1, then there is in g a maximal regular Abelian subgroup h containing h_1.

                    M. S. Knebelman (Pullman, Wash.).

Siegel, Carl Ludwig. Discontinuous groups. Ann. of Math. (2) **44**, 674–689 (1943). [MF 9405]

Let G be a topological group and H a discrete subgroup. Then a subset F of G is called a fundamental set (relative to H) if (1) $FH = G$, (2) $Fa \cap F = 0$ for every a in H different from the identity and (3) F is a Borel set. The author shows that fundamental sets exist for any separable group G. An explicit construction is given. A fundamental set F is normal if every point in G has a neighborhood contained in the union of a finite number of images Fa, a in H; Fa is called a neighbor of F if the closure $\bar{F} \cap \bar{Fa} \neq 0$. It is shown that, if G is separable and connected and F is a normal

fundamental set, then H is generated by the elements a such that Fa is a neighbor of F. Thus, if F has only a finite number of neighbors, H is finitely generated. If G is locally compact and separable and v denotes the Haar measure, then it is shown that $v(F)$, for F a fundamental set, does not depend on F but only on the subgroup H; H is said to be a subgroup of the first kind in G if there exists a normal fundamental set F of finite measure having only a finite number of neighbors.

In the second part of the paper the author proves that the group H of units of an order of a simple algebra A_0 over the rational field is of the first kind in G, the group of matrices of norm ± 1 in $A = A_{0R}$, R the real field. This implies a result previously proved by Eichler for A_0 a division algebra, namely, that H is finitely generated. The method of proof of this theorem consists in studying the discontinuous representation of H in a coset space C/G of G with respect to a suitable compact group C. In the general case of an arbitrary separable locally compact group, the author proves that, if $v(F)$ is finite, then the representation of the discrete group H in C/G, C a closed subgroup, is discontinuous if and only if C is compact.

N. Jacobson (Baltimore, Md.).

Kowalewski, Gerhard. Neues Beispiel einer genetisch darstellbaren Berührungstransformation. J. Reine Angew. Math. **185**, 102–105 (1943). [MF 9591]

An example of a one parameter group of real contact transformations. *C. Chevalley* (Princeton, N. J.).

Stoll, R. R. Representations of finite simple semigroups. Duke Math. J. **11**, 251–265 (1944). [MF 10664]

The fundamental problem of this paper is to represent a finite semigroup S as a set P of mappings of a set N into itself. This can always be done with the number of points in N equal to the number of elements in S. A "true" representation (the representation isomorphic to S) can always be made using one more point. It is shown that N can be decomposed uniquely into a minimal number of sets on each of which P is transitive. Consequently such a representation can be regarded as a collection of semigroups on transitive systems.

If S is simple and has no zero, then stronger theorems can be stated. D. Rees [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127] has given a different type of representation for these and has shown that they are the union of isomorphic groups. Using the results of Rees, the author shows that all other representations by means of mappings can be constructed from the transitive ones. An interesting construction is given in terms of the cosets of S.

H. Campaigne (Washington, D. C.).

Albert, A. A. Quasigroups. I. Trans. Amer. Math. Soc. **54**, 507–519 (1943). [MF 9523]

The author introduces a new method for the study of quasigroups. A set $\mathfrak{G}$ is a quasigroup when a unique product $a \cdot b$ is defined such that the equations $ax = b$, $ya = b$ have unique solutions. A quasigroup with a unit is called a loop.

The right and left multiplications by elements x define sets of permutations $\{R_x\}$ and $\{L_x\}$. Conversely, a quasigroup may be defined by a set of permutations $x \rightleftarrows R_x$ such that two distinct permutations give distinct results for all elements in $\mathfrak{G}$. Two quasigroups are isotopic when they consist of the same elements and one can find permutations A, B, C such that their right multiplications are related by the rule $R_x^{(0)} = A R_{xB} C$. When $C = 1$, one has a principal isotope. Isomorphism is a special case of isotopy and every isotope is isomorphic to a principal isotope. Each quasigroup has a principal isotope which is a loop. A loop isotopic to a group is an isomorphic group.

With every subset $\mathfrak{H}$ of a quasigroup there are associated groups $\mathfrak{H}_r$, $\mathfrak{H}_\lambda$, $\mathfrak{H}_r$, generated by the right, left, all multiplicative permutations defined by the elements in $\mathfrak{H}$. In this manner the properties of a quasigroup $\mathfrak{G}$ are projected on those of its associated group $\mathfrak{G}_r$. For a subloop $\mathfrak{H}$ the (right) cosets are defined to be the sets $x\mathfrak{H}_r$. Normal subloops are associated with normal subgroups in $\mathfrak{G}_r$, and the ordinary homomorphism properties hold. A subloop is simple when it has no normal subloop and simple loops have simple isotopes. Certain other topics analogous to group concepts are also discussed. *O. Ore.*

Bruck, Richard H. Some results in the theory of quasigroups. Trans. Amer. Math. Soc. **55**, 19–52 (1944). [MF 9873]

Let G be a quasigroup in which the product of any two elements a and b is denoted by $a \cdot b$ and let G_0 be a quasigroup consisting of the same elements as G, the product of a and b in G_0 being represented by $a \circ b$. Then G_0 is isotopic to G if there exist three one-to-one reversible mappings U, V and W of G into itself such that $a \circ b = (a^u \cdot b^v)^w$. The concept of isotopy is used to unify much of the known theory of quasigroups and to obtain many new results.

Every quasigroup is isotopic to one with a unique two sided unit element. A quasigroup Q is said to have the inverse property (I.P. quasigroups) if there exist two one to one reversible mappings L and R of Q on itself such that $a^L(ab) = (ba)a^R = b$ for all a, b in Q. All I.P. quasigroups isotopic to a group are determined. Quasigroups constructed from nonassociative algebras by use of the multiplication rule $x \circ y = x + y + xy$ are studied. A quasigroup with unique two sided unit which satisfies the associative law $a(b \cdot cb) = (ab \cdot c)b$ is called a Moufang quasigroup. If Q has a unit element the necessary and sufficient condition that every isotope of Q with unit element be an I.P. quasigroup is that Q be Moufang. It follows that, just as every quasigroup with unit and isotopic to a group is itself a group, so every quasigroup with unit and isotopic to a Moufang quasigroup is itself a Moufang quasigroup. The Abelian quasigroups previously discussed by the reviewer are also considered, and a new explicit construction for all such quasigroups is given. Various other special types of quasigroups are dealt with and necessary and sufficient conditions are given that a direct product of quasigroups contain no proper subquasigroup. *D. C. Murdoch.*

ANALYSIS

Tamarkin, J. D. and Zygmund, A. Proof of a theorem of Thorin. Bull. Amer. Math. Soc. **50**, 279–282 (1944). [MF 10212]

The central theorem, which is a generalization of a result of M. Riesz, is as follows. (i) Let $f(z_1, \cdots, z_r)$ be an entire function of r complex variables $z_1, \cdots, z_r$. Let K be a bounded domain $(v_1, \cdots, v_r)$ of the r-dimensional Euclidean space, satisfying the conditions $v_1 \geqq 0, \cdots, v_r \geqq 0$. Let $M(\alpha_1, \cdots, \alpha_r)$ denote the upper bound of $|f(z_1, \cdots, z_r)|$ for $|z_1| = v_1^{\alpha_1}, \cdots, |z_r| = v_r^{\alpha_r}$, $(v_1, \cdots, v_r) \epsilon K$. Then

$\log M(\alpha_1, \cdots, \alpha_r)$ is a convex function of the point $(\alpha_1, \cdots, \alpha_r)$ in the domain $0 \leq \alpha_j < \infty$; $j=1, 2, \cdots, r$. (ii) If the points of K satisfy a condition $0 < A \leq v_j \leq B < \infty$, then $\log M(\alpha_1, \cdots, \alpha_r)$ is convex in the whole space $-\infty < \alpha_j < \infty$.

The proof depends on the principle that the logarithm of a positive function $\phi(t)$ is convex in an interval if for every real μ the function $\phi(t)e^{\mu t}$ has no proper maximum in the interval. It is shown that, if $M(\alpha_1, \cdots, \alpha_r)e^{\mu t}$, where the α_i are linear functions of t, has a proper maximum, then $|f(z_1, \cdots, z_r)e^{\mu t}|$, where the z_j are certain regular functions of t, does also, even for complex t, in contradiction to the maximum principle of regular functions. Part (ii) is a limiting case of (i). Thus a simple proof of the theorem is obtained. Related results are discussed.   *A. C. Schaeffer.*

Bellman, Richard. A note on periodic functions and their derivatives. J. London Math. Soc. 18, 140–142 (1943). [MF 10364]

Let $f(x)$ be a function of period 2π, having $k-1$ derivatives, the last of which is absolutely continuous. Let also $\int_0^{2\pi} f \, dx = 0$. Then

$$(*) \quad \left\{ \int_0^{2\pi} |f(x)|^{2r} dx \right\}^{1/2r} \leq a_k \left\{ \int_0^{2\pi} |f^{(k)}(x)|^{2r} dx \right\}^{1/2r},$$
$$r = 1, 2, \cdots,$$

where

$$a_k = 4\pi^{-1} \sum_0^\infty (2\nu+1)^{-(k+1)}, \quad a_k = 4\pi^{-1} \sum_0^\infty (-1)^\nu (2\nu+1)^{-(k+1)}$$

according as k is odd or even. The proof follows the pattern of the case $r = \infty$ considered earlier by Northcott. [J. London Math. Soc. **14**, 198–202 (1939); these Rev. **1**, 71. More general results will be found in the Russian book by N. Achyeser, Lectures on the Theory of Approximation [Kharkoff, 1940; these Rev. **3**, 234]. See also J. Favard, Bull. Sci. Math. (2) **61**, 209–224, 243–256 (1937) and N. Achyeser and M. Krein. C. R. (Doklady) Acad. Sci. USSR (N.S.) **15**, 107–111 (1937).]
    A. Zygmund (South Hadley, Mass.).

Zygmund, A. On certain integrals. Trans. Amer. Math. Soc. 55, 170–204 (1944). [MF 10214]

The paper is devoted to the fine points of a theory of inequalities valid for integrals of the form

$$\mathfrak{J}_r[f] = \left\{ \int_0^{2\pi} |f(\theta)|^r d\theta \right\}^{1/r}$$

which is important in the theory of trigonometric series and of related harmonic and analytic functions. It is divided into three parts. Let $f(r, \theta)$ be the harmonic function given by the Poisson integral of $f(\theta)$, which is the real part of an analytic function $\phi(z)$, and

$$[g^*(\theta)]^2 = \pi^{-1} \int_0^1 (1-\rho) \int_0^{2\pi} |\phi'(\rho e^{i(\theta+t)})|^2 P(\rho, t) dt,$$

where $P(\rho, t)$ is the Poisson kernel. Then the first part of the paper consists of the proof of the following inequality: $A_r \mathfrak{J}_r(f) \leq \mathfrak{J}_r(g^*) \leq B_r \mathfrak{J}_r(f)$, where $r > 1$ and A_r, B_r denote certain constants depending only on r. The inequality is due to Littlewood and Paley [Proc. London Math. Soc. (2) **42**, 52–89 (1936)] and was proved by them for all positive even integers only. The second part of the paper proves the theorem: if

$$F(t) = \int_0^t f(u) du,$$

$$[\mu_r(\theta)]^r = \int_0^\pi |F(\theta+t) + F(\theta-t) - 2F(t)|^r / t^{r+1} \cdot dt$$

and $f(u)\epsilon L^r$ and $F(u)$ have the period 2π, then there are two positive constants A_r, B_r depending only on the subscripts such that $A_r \mathfrak{J}_r[f] \leq \mathfrak{J}_r[\mu_r] \leq B_r \mathfrak{J}_r[f]$. The proof is long and intricate and has to use the detour over the complex domain. The third part contains the proof of the following result: if $\sum c_n$ is lacunary and absolutely Abel summable (that is, $\sum c_n r^n$ is of bounded variation), then it is absolutely convergent. The corresponding result without the "absolute" is Hardy-Littlewood's high indices theorem. This new theorem has interesting corollaries for harmonic functions which enable the author to throw new sidelights on the way an analytic function in $|z| < 1$ assumes its boundary values.   *František Wolf* (Berkeley, Calif.).

Kharadze, A. K. Sur l'identité d'Euler-Lagrange et de l'inégalité de Bouniakowski-Schwarz. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 1–8 (1942). (Russian. Georgian and French summaries) [MF 10311]

The author obtains a generalization of the identity

$$\frac{1}{2} \int_a^b \int_a^b \begin{vmatrix} f(x) & f(y) \\ \phi(x) & \phi(y) \end{vmatrix}^2 dx dy = \begin{vmatrix} \int_a^b f^2 dx & \int_a^b f\phi dx \\ \int_a^b \phi f dx & \int_a^b \phi^2 dx \end{vmatrix},$$

when the number of functions involved is three or four, by the introduction of new algebraic expressions called quasi-determinants. Let $a, b, \cdots, k$ be m vectors of dimension m and $a_1 b_2 \cdots k_m$ be the product of the elements in the main diagonal of the square matrix $A = (ab \cdots k)$. Let P denote the cyclic permutation $(12 \cdots m)$ and ω a primitive mth root of unity. Then

$$\sum_{i=0}^{m-1} \omega^i P^i(a_1 b_2 \cdots k_m)$$

is called the quasi-determinant of A and is denoted by $[a_1 b_2 \cdots k_m]$. Thus, if $m=2$, the quasi-determinant of A is the ordinary determinant of A and, if $m=3$,

$$[a_1 b_2 c_3] = a_1 b_2 c_3 + \omega a_2 b_3 c_1 + \omega^2 a_3 b_1 c_2,$$

while

$$[a_1 b_2 c_k] = a_1 b_2 c_k + \omega a_2 b_k c_i + \omega^2 a_k b_i c_j,$$

$\omega^3 = 1$, $\omega \neq 1$. The author then proves the two identities

$$\frac{1}{3} \sum_{i,j,k} |[a_i b_j c_k]|^2 = \sum_{r=1}^n a_r^2 \sum_{r=1}^n b_r^2 \sum_{r=1}^n c_r^2 - \sum_{r=1}^n a_r b_r \sum_{r=1}^n b_r c_r \sum_{r=1}^n c_r a_r$$

and

$$\frac{1}{4} \sum_{i,j,k,l} |[a_i b_j c_k d_l]|^2 = \sum_{r=1}^n a_r^2 \sum_{r=1}^n b_r^2 \sum_{r=1}^n c_r^2 \sum_{r=1}^n d_r^2$$
$$- (\sum_{r=1}^n a_r c_r)^2 (\sum_{r=1}^n b_r d_r)^2,$$

from which the desired identities

$$\frac{1}{3} \int_a^b \int_a^b \int_a^b |[f(x)\phi(y)\psi(z)]|^2 dx dy dz$$
$$= \int_a^b f^2 dx \int_a^b \phi^2 dx \int_a^b \psi^2 dx - \int_a^b f\phi dx \int_a^b \phi\psi dx \int_a^b \psi f dx$$

and

$$\frac{1}{4} \int_a^b \int_a^b \int_a^b \int_a^b |[f(x)\phi(y)\psi(z)\omega(t)]|^2 dx dy dz dt$$
$$= \int_a^b f^2 dx \int_a^b \phi^2 dx \int_a^b \psi^2 dx \int_a^b \omega^2 dx - \left(\int_a^b f\psi dx \right)^2 \left(\int_a^b \phi\omega dx \right)^2$$

are deduced. In each of these last two identities appears the quasi-determinant of the corresponding matrix of Gram.

J. Williamson (Flushing, N. Y.).

Combes, Bernard. **Sur les développements en série du type de Taylor.** C. R. Acad. Sci. Paris **216**, 281–283 (1943). [MF 10013]

The author considers the convex set D consisting of the centers of mass of a positive mass distribution df on a curve C in n-space, where f belongs to a class Δ of functions defined by positive linear inequalities. [See the author's preceding note, C. R. Acad. Sci. Paris **215**, 291–293 (1942); these Rev. **5**, 106.] The points of D can be obtained as linear combinations with positive coefficients of certain boundary points, called "vertices" of D. That corresponds to expansions of any function f of Δ in terms of "limiting" functions. As a special case Taylor expansions are obtained. No details are given.   *F. John* (Aberdeen, Md.).

Theory of Sets, Theory of Functions of Real Variables

Cuesta Dutari, Norberto. **Decimal theory of the order types.** Revista Mat. Hisp.-Amer. (4) **3**, 186–205, 242–268 (1943). (Spanish) [MF 10139]

A "generalized dyadic decimal" is a transfinite sequence of the following type: $A_\xi = 0. a_0 a_1 \cdots a_\nu \cdots$, where a_ν is 0 or 1 and $0 \leqq \nu < \xi$, ξ being any fixed ordinal called the index of the number. They are ordered as follows. (1) If A_ξ and B_η are two such numbers with $\xi < \eta$ and $a_\nu = b_\nu$ for $0 \leqq \nu < \xi$, then $A_\xi < B_\eta$ or $A_\xi > B_\eta$ according as b_ξ is 1 or 0 (for example, $0.1101 < 0.11 < 0.1110$). (2) If $a_\nu \neq b_\nu$ for some $\nu < \min(\xi, \eta)$, let ρ be the first such ordinal. Then $A_\xi < B_\eta$ or $A_\xi > B_\eta$ according as $a_\rho = 0$ and $b_\rho = 1$ or $a_\rho = 1$ and $b_\rho = 0$ (for example, $0.0010 < 0.010$). Consequently two such numbers are equal if and only if their sequences are identical. It is shown that every order type whose cardinal number is $|\omega_\iota|$ can be realized by a set of "generalized dyadic decimal numbers" of index less than ω_ι. The structure of order types is investigated by means of such realizations, and particular properties of the structure are characterized by means of the "decimal representation." By this means the author is also able to obtain new proofs and generalizations of known theorems.   *J. V. Wehausen.*

Day, Mahlon M. **Oriented systems.** Duke Math. J. **11**, 201–229 (1944). [MF 10159]

An oriented system S is a nonempty set with a transitive binary relation $>$ such that every element has a successor. If each pair of elements has a common successor the system is directed. A subsystem T of S is called cofinal in S if every element of S has a successor in T. Two oriented systems are called cofinally equivalent if they are isomorphic with cofinal subsystems of a third oriented system. In addition to this equivalence relation the author studies in detail two different ways of introducing a partial order in the totality of oriented systems. The maximal directed subsystems of S are studied. If these maximal directed subsystems are pairwise disjoint, S is called "simple." If every element of S has a pair of successors without a common successor, S is called "everywhere branching." The classification of the cofinal equivalence classes is carried out for countable oriented systems.   *S. Eilenberg* (Ann Arbor, Mich.).

Zorn, Max. **Idempotency of infinite cardinals.** Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 9–12 (1944). [MF 10452]

The theorem which states that an infinite cardinal number is equal to its square is usually derived by means of ordinal numbers. This paper presents a proof without ordinals, based on the author's well-known maximum principle.   *L. H. Loomis* (Cambridge, Mass.).

Vázquez García, Roberto and Zubieta Russi, Francisco. **The linear homogeneous continua of George D. Birkhoff.** Bol. Soc. Mat. Mexicana **1**, No. 2, 1–14 (1944). (Spanish) [MF 10557]

The authors consider an ordered set of elements H satisfying the following axioms. (1) If $A \varepsilon H$, there exist $P, Q \varepsilon H$ such that $P < A < Q$. (2) For any sequence of the type $\cdots < A_{-2} < A_{-1} < A_0 < A_1 < \cdots$, the elements A_n together with the elements $X \varepsilon H$ such that $A_n < X < A_{n+1}$, $n = 0, \pm 1, \pm 2, \cdots$, form an "interval." (3) Between any two intervals it is possible to establish a one-to-one order-preserving correspondence. The term "interval" in (2) and (3) denotes any of the following sets: all $X \varepsilon H$ such that $A < X < B$, all X such that $X > A$, all X such that $X < A$, the whole set H. Various properties of sets satisfying these axioms (linear homogeneous continua) are established. Another axiom is then introduced. (4) Every set of disjoint intervals is denumerable. It is shown that, if a set satisfies these four axioms, there is an order-preserving one-to-one correspondence between it and the continuum of real numbers. An example is given of a set satisfying (1), (2) and (3) but not (4).   *J. V. Wehausen* (Columbia, Mo.).

Zubieta Russi, Francisco and Vázquez García, Roberto. **Note on the continuum.** Bol. Soc. Mat. Mexicana **1**, No. 2, 15–17 (1944). (Spanish) [MF 10586]

It is shown that among ordered sets the continuum of real numbers is characterized, up to a one-to-one order-preserving correspondence, by the following four axioms. (1) For any a there exist b and c such that $b < a < c$. (2) For any a and b, $a < b$, there exists c such that $a < c < b$. (3) Every infinite increasing sequence is either convergent (defined in terms of order) or else eventually surpasses any given element. (4) Every set of disjoint intervals [defined as in the preceding review] is denumerable. According to the authors the problem of demonstrating this result was proposed by Souslin.   *J. V. Wehausen* (Columbia, Mo.).

Morse, Anthony P. **A theory of covering and differentiation.** Trans. Amer. Math. Soc. **55**, 205–235 (1944). [MF 10215]

Let $\mathfrak{S}$ be a metric space and let ϕ be a measure in $\mathfrak{S}$. If Δ is a nonnegative function whose domain is a family $\mathfrak{H}$ of sets in $\mathfrak{S}$, then, for each β in $\mathfrak{H}$, define $\Delta : \beta$ as the set of elements x in $\mathfrak{S}$ for which there is a set α in $\mathfrak{H}$ such that $x \varepsilon \alpha$, $\beta \alpha \neq 0$ and $\Delta(\alpha) \leqq 2\Delta(\beta)$. Basic principle [theorem 3.10]: if $\mathfrak{F}$ covers A in the sense of Vitali and Δ is a nonnegative function which is bounded on $\mathfrak{F}$, then $\mathfrak{F}$ contains a disjointed family $\mathfrak{G}$ for which

$$A - \sum_{\beta \varepsilon \mathfrak{G}} \beta \subset \sum_{B \varepsilon \mathfrak{H}} \Delta : B$$

whenever $\mathfrak{H} \subset \mathfrak{G}$ and $\mathfrak{G} - \mathfrak{H}$ is finite. A comprehensive form of the Vitali covering theory is based on this principle. A blanket is a function F such that, for x in its domain: (i) $x \varepsilon \mathfrak{S}$ and $F(x)$ is a family of nonvacuous subsets of $\mathfrak{S}$;

(ii) $\beta\epsilon F(x)$ implies diam $\beta < \infty$; (iii) $\inf_{\beta\epsilon F(x)}$ diam $(\beta + \{x\})$ $=0$. Various types of blankets are defined, and many of their properties are developed. In particular, the idea of a regular blanket is formulated in terms of the measure ϕ and the function Δ so as to lead to a general abstract differential theory of nonadditive functions.

C. C. Torrance (Cleveland, Ohio).

Morse, A. P. and Randolph, John F. The ϕ rectifiable subsets of the plane. Trans. Amer. Math. Soc. 55, 236–305 (1944). [MF 10216]

Let ϕ be a measure, and L Carathéodory linear measure, in two-dimensional Euclidean space E_2. Let $\mathfrak{R}_x{}^r$ denote the open circle with center $x = (x_1, x_2)$ and radius r. For $x\epsilon E_2$ and $A \subset E_2$, let

$$\mathfrak{D}_\phi{}^\triangledown(A, x) = \lim_{r\to0+} \frac{\phi(A\mathfrak{R}_x{}^r)}{2r}, \quad \mathfrak{D}_\phi{}^\triangle(A, x) = \overline{\lim_{r\to0+}} \frac{\phi(A\mathfrak{R}_x{}^r)}{2r},$$

$$\mathfrak{D}_\phi{}^\blacktriangle(A, x) = \overline{\lim_{(\text{diam } \Gamma)\to0, r\epsilon\mathfrak{H}}} \frac{\phi(A\Gamma)}{(\text{diam } \Gamma)},$$

where $\mathfrak{H}$ is the family of circles Γ containing x. A set $A \subset E_2$ is strictly rectifiable if and only if there exist a number $M > 0$, a closed interval J and a function f, with domain J and range A, such that f satisfies the Lipschitz condition

$$|f(t'') - f(t')| \leq M |t'' - t'|$$

for t' and t'' in J. A set A is rectifiable if and only if A is a subset of a strictly rectifiable set. A set A is ϕ rectifiable if and only if to each $\epsilon > 0$ there exists a strictly rectifiable set B with $\phi(A\bar{B}) < \epsilon$, where $\bar{B}$ is the complement of B. For $x\epsilon E_2$ and $A \subset E_2$, let (i) Sgn (A, x) denote the closure of the set of points $(y-x)/|y-x|$ for $y\epsilon A - \{x\}$; (ii) dir (A, x) $= \prod$ Sgn $(A\bar{\beta}, x)$ taken over all sets β such that $\beta\mathfrak{R}_x{}^r = 0$ for some $r > 0$; (iii) $\text{dir}_\phi (A, x) = \prod$ Sgn $(A\bar{\beta}, x)$ taken over all sets β such that $\mathfrak{D}_\phi{}^\triangle(\beta, x) = 0$. [dir (A, x) is also denoted by $\text{contg}_\triangle x$.] A set is diametral if and only if it is of the form $\{-z\} + \{z\}$, where $z\epsilon E_2$ and $|z| = 1$. A set $A \subset E_2$ is directional at x if and only if dir (A, x) is dimetral. A set A is ϕ directional at x if and only if $\text{dir}_\phi (A, x)$ is diametral. A set A is restricted at x if and only if there exists a point $z\epsilon E_2$ with $|z| = 1$ for which $\{z\}$ dir $(A, x) = 0$. A set A is ϕ restricted at x if and only if $\mathfrak{D}_\phi{}^\triangle(A, x) = 0$ and there exists a diametral set Z for which $Z \, \text{dir}_\phi (A, x) = 0$.

Summary theorem 11.1. If $A \subset E_2$, $\phi(A) < \infty$, $\mathfrak{D}_\phi{}^\triangledown(A, x)$ $< \infty$ for ϕ almost all x in A, then the following nine propositions are equivalent. (1) A is ϕ rectifiable. (2) $\mathfrak{D}_\phi{}^\triangle(A, x)$ $<1.01\mathfrak{D}_\phi{}^\triangledown(A, x)$ for ϕ almost all x in A. (3) $3\mathfrak{D}_\phi{}^\blacktriangle(A, x)$ $<4\mathfrak{D}_\phi{}^\triangledown(A, x)$ for ϕ almost all x in A. (4) $0 < \mathfrak{D}_\phi{}^\triangledown(A, x)$ $=\mathfrak{D}_\phi{}^\triangle(A, x) < \infty$ for ϕ almost all x in A. (5) $0 < \mathfrak{D}_\phi{}^\triangledown(A, x)$ $=D_\phi{}^\blacktriangle(A, x) < \infty$ for ϕ almost all x in A. (6) A is ϕ directional at ϕ almost all of its points. (7) A is ϕ restricted at ϕ almost all of its points. (8) To each $\epsilon > 0$ there exists a set $B \subset A$ such that $\phi(A\bar{B}) < \epsilon$ and such that B is directional at each of its points. (9) To each $\epsilon > 0$ there exists a set $B \subset A$ such that $\phi(A\bar{B}) < \epsilon$ and such that B is restricted at each of its points. If ϕ is replaced by L in this theorem, then this theorem is essentially a compendium of results of A. S. Besicovitch [Math. Ann. 115, 296–329 (1938)]. Theorem 11.6. If $A \subset E_2$, $\phi(A) < \infty$, $L(A) < \infty$, $\mathfrak{D}_\phi{}^\triangle(A, x) > 0$ for L almost all x in A, $\mathfrak{D}_\phi{}^\triangle(A, x) < \infty$ for ϕ almost all x in A, then A is ϕ rectifiable if and only if A is L rectifiable. This paper includes compendia of general properties of measure and density.   C. C. Torrance.

Loomis, Lynn H. The intrinsic measure theory of Riemannian and Euclidean metric spaces. Ann. of Math. (2) 45, 367–374 (1944). [MF 10274]

Let M be a boundedly compact metric space with the property that there exists a constant $K \geq 1$ such that, if some closed sphere of radius r can be covered by n open spheres of radius x, then, for any positive a, every closed sphere of radius ar can be covered by n open spheres of radius Kax. Let μ denote Hausdorff α-dimensional measure. Principal result: if S is an open sphere with radius r, then

$$K^{-2\alpha}r^\alpha \leq \mu(S) \leq K^{2\alpha}r^\alpha.$$

In particular, $\mu(S) = r^\alpha$ when $K = 1$ (Euclidean metric space). The author extends this result to other kinds of spaces and considers the uniqueness of volume measure in these spaces.   C. C. Torrance (Cleveland, Ohio).

Rickart, C. E. Decomposition of additive set functions. Duke Math. J. 10, 653–665 (1943). [MF 9748]

The set functions considered have their domains in an hereditary subset of a σ-field $\mathfrak{M}$ (of subsets of an abstract set M) and their ranges in a linear normed vector space $\mathfrak{X}$. A function x is said to be strongly bounded (or s-bounded) in a set $\mathfrak{A}$ provided $\lim_{n\to\infty} x(e_n) = 0$ for disjointed $e_n\epsilon\mathfrak{A}$. The following theorem, indicative of the results of the paper, is a generalization of a result of R. S. Phillips [Bull. Amer. Math. Soc. 46, 274–277 (1940); these Rev. 1, 240]. Let x be an additive function which is s-bounded and let $\mathfrak{A}$ be a σ-ideal contained in $\mathfrak{D}$, the domain of x. Then $x(e)$ can be decomposed uniquely in the form $x(e) = x^0(e) + x^1(e)$, where x^0 is additive, defined over $\mathfrak{D}$ and vanishes in $\mathfrak{A}$, and x^1 is additive, defined over all of $\mathfrak{M}$ and vanishes outside a set $e^0\epsilon\mathfrak{A}$ (that is, $x^1(e) = x^1(e \cap e^0)$).

The following considerable extension of the Lebesgue decomposition theorem is also given. Let x be a completely additive function over $\mathfrak{M}$ and let u be an outer measure function over $\mathfrak{M}$ (that is, u is a monotone increasing convex function on $\mathfrak{M}$ to the extended nonnegative real number system). Then $x(e)$ can be decomposed uniquely into the form $x(e) = x^0(e) + x^1(e)$ for $e\epsilon\mathfrak{M}$, where x^0 and x^1 are completely additive functions over $\mathfrak{M}$, x^0 is absolutely continuous relative to u and there exists a set $e^0\epsilon\mathfrak{M}$ such that $u(e^0)$ and $x^1(e) = x^1(e \cap e^0)$ for $e\epsilon\mathfrak{M}$.   J. F. Randolph.

Franklin, Philip. Measurable functions. J. Math. Phys. Mass. Inst. Tech. 23, 24–44 (1944). [MF 10116]

The paper is an expository article on the Lebesgue equivalence of measurable functions with Baire functions of class two. The author also shows that a Besicovitch almost periodic function is Besicovitch equivalent with a function having derivatives of all orders.   S. Bochner.

Singh, A. N. On functions without one-sided derivatives. II. Proc. Benares Math. Soc. (N.S.) 4, 95–108 (1943). [MF 10355]

[The first part appeared in the same Proc. 3, 55–69 (1941); these Rev. 5, 175.] This note gives a general method of constructing a class of functions which do not possess a progressive or regressive derivative at any point. Such a function has been constructed geometrically by Besicovitch, and in part I of this paper the author gave an analytical definition of this function. The methods of the present part II are simpler and more elegant, but they are not suitable to be given in a brief review.   R. L. Jeffery.

Shukla, Uma Kant. **On a non-differentiable function.** Proc. Benares Math. Soc. (N.S.) **4**, 71–76 (1943). [MF 10353]

In this note a nondifferentiable function $f(x)$ is defined on the interval $(0, 1)$ by changing digits in the decimal representation of the variable x. These rules involve a table which is not suitable for insertion in this review. It is shown that the function is continuous; also, at every point except possibly $x = 0.999 \cdots$, one of the median derivatives is zero, while one of the extreme derivatives is either ∞ or $-\infty$. *R. L. Jeffery* (Wolfville, N. S.).

Ostrowski, Alexandre. **Note sur l'interversion des dérivations et les différentielles totales.** Comment. Math. Helv. **15**, 222–226 (1943).

Defining uniform differentiability of a function $f(x_1, \cdots, x_n)$ with respect to x_r at a point $P(a_1, \cdots, a_n)$ as the existence of the limit

$$\frac{f(x_1, \cdots, x_{r-1}, x_r, \cdots, x_n) - f(x_1, \cdots, x_{r-1}, a_r, \cdots, x_n)}{x_r - a_r}$$

as $x_r - a_r \to 0$, while $|x_i - a_i| \leq |x_r - a_r|$, $i = 1, \cdots, n$, the author proves: (1) uniform differentiability of $f(x_1, \cdots, x_n)$ with respect to $x_1, \cdots, x_n$ at P is necessary and sufficient for the existence of a total differential of $f(x_1, \cdots, x_n)$ at P in the usual sense; (2) if $f(x_1, x_2)$ possesses first partial derivatives f_1 and f_2 at $P(a_1, a_2)$ and if, at P, f_1 has a uniform partial derivative with respect to x_2, and f_2 a uniform partial derivative with respect to x_1, then $f_{12} = f_{21}$. He also gives an example to show that, if the condition of uniform differentiability is weakened by replacing the condition $|x_i - a_i| \leq |x_r - a_r|$ by $|x_i - a_i| \leq (1 - \epsilon)|x_r - a_r|$, $i \neq r$, for a fixed positive ϵ, then the result stated in (2) does not hold. *A. Dresden* (Swarthmore, Pa.).

Nef, Walter. **Über die Stieltjes'schen Integrale.** Comment. Math. Helv. **16**, 29–36 (1944).

A complicated proof of the well-known theorem that

$$\int_a^b f(x) d\alpha_1(x) = \int_a^b f(x) d\alpha_2(x)$$

for all continuous functions $f(x)$ with $\alpha_1(x)$ and $\alpha_2(x)$ of bounded variation if and only if $\alpha_1(x) - \alpha_2(x)$ differs from a constant at most at a denumerable set interior to (a, b). *T. H. Hildebrandt* (Ann Arbor, Mich.).

Čelidze, V. G. **On the representation of functions of two variables by singular double integrals.** Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 1–22 (1942). (Russian. Georgian summary) [MF 10280]

Let $\varphi_{mn}(x, y; \alpha, \beta)$, $m, n = 1, 2, \cdots$, be a double sequence of measurable functions defined for $a_1 < x < b_1$, $a_2 < y < b_2$, $a_1 \leq \alpha \leq b_1$, $a_2 \leq \beta \leq b_2$ and satisfying

$$\lim_{(m,n)_k \to \infty} \int_{\lambda_1}^{\lambda_2} \int_{\mu_1}^{\mu_2} \varphi_{mn}(x, y; \alpha, \beta) d\alpha d\beta = 1$$

for any $\lambda_1, \lambda_2, \mu_1, \mu_2$ with $a_1 \leq \lambda_1 < x < \lambda_2 \leq b_1$ and $a_2 \leq \mu_1 < y < \mu_2 \leq b_2$, where by the symbol $(m, n)_k$ is meant a pair (m, n) such that $1/k \leq m/n \leq k$, k a positive integer. The integral

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi_{mn}(x, y; \alpha, \beta) f(\alpha, \beta) d\alpha d\beta,$$

where $f(\alpha, \beta)$ is defined on $[a_1, b_1; a_2, b_2]$, is called a singular double integral. Several theorems lead up to the following main theorem. In order that, for any measurable and

bounded $f(\alpha, \beta)$,

$$\lim_{(m,n)_k \to \infty} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi_{mn}(x, y; \alpha, \beta) f(\alpha, \beta) d\alpha d\beta = f(x, y)$$

at any point (x, y) of approximate continuity of $f(\alpha, \beta)$, it is necessary and sufficient that $\varphi_{mn}(x, y; \alpha, \beta)$ satisfy the following condition:

$$\lim_{(m,n)_k \to \infty} \int \int_E \varphi_{mn}(x, y; \alpha, \beta) d\alpha d\beta = 1$$

for any measurable set $E \subset [a_1, b_1; a_2, b_2]$ for which (x, y) is a point of density. [For definitions of "approximate continuity" and "point of density" we refer to Saks, Theory of the Integral, Warsaw, 1937, pp. 131, 132 and 128, respectively.] The proof is developed in a manner similar to that given by Natanson for the one-dimensional case [Fund. Math. **18**, 99–109 (1932)]. *J. V. Wehausen.*

Youngs, J. W. T. **Curves and surfaces.** Amer. Math. Monthly **51**, 1–11 (1944). [MF 10063]

Let A denote a metric space and suppose $f(a) = b$ is a continuous transformation from A into another metric space. A transformation $f_1(A_1)$ is said to be equivalent to $f_2(A_2)$ if and only if there is, for every $\epsilon > 0$, a topological transformation $T_\epsilon(A_1) = A_2$ such that $\rho\{f_1(a_1), f_2(T_\epsilon(a_1))\} < \epsilon$ for every a_1 in A_1. The equivalence class $[f]$ generated by f over a base space A is called a curve or surface if A is a topological curve or surface. The author gives a brief expository discussion of the ideas of arc-length, surface-area and graph from the point of view of the preceding definitions. *C. C. Torrance* (Cleveland, Ohio).

Theory of Functions of Complex Variables

Augé, Juan. **On the zeros of polynomials and Laurent series.** Revista Mat. Hisp.-Amer. (4) **3**, 176–185, 229–241 (1943). (Spanish) [MF 10138]

The first part gives inequalities, in terms of the absolute values for the coefficients, for circles and circular rings containing or not containing zeros of polynomials and Laurent series. These make more precise, in various ways, results of Ostrowski [Acta Math. **72**, 99–257 (1940); these Rev. **1**, 323; **2**, 342] and San Juan [Revista Mat. Hisp.-Amer. (3) **1**, 1–14 (1939); these Rev. **2**, 61]. The second part gives improved estimates for the error in the roots introduced by taking only a part of the series, or by approximating its coefficients. *R. P. Boas, Jr.* (Cambridge, Mass.).

Calugareanu, Georges. **Singularités des fonctions analytiques uniformes et polynomes de Tchebichef.** Mathematica, Timişoara **19**, 139–147 (1943). [MF 9961]

A single-valued function $f(z)$ analytic at infinity is given by $f(z) = \sum_0^\infty \lambda_n z^{-n}$. It is proposed to determine its singularities, the closed set S, with the aid of certain sequences of Tchebychef polynomials. Consider a quadratic lattice of span r and let $S(r)$ be the set of all closed squares containing points of S. Let $T_n(z)$ and $T_n(z, r)$ be the nth Tchebychef polynomials attached to S and $S(r)$, respectively. The author puts two problems: (i) to prove that the zeros of $T_n(z)$ tend to S when $n \to \infty$ and (ii) to determine $T_n(z)$ as a functional of $f(z)$. In connection with (i) he proves that, by a limit relation due to R. Nevanlinna, $T_n(z, r)$ can have no zeros outside of $S(r)$ for large n.

Furthermore, $\lim_{n\to\infty} T_n(z, r2^{-n}) = T_n(z)$ uniformly in $|z| \leq R$ for fixed n, whence it follows that no fixed regular point of $f(z)$ can be a zero of $T_n(z)$ for large n. Every isolated point of S is a limit point of zeros of $T_n(z)$, but, if S is, for instance, an elliptical region, then only the points on the line segment joining the foci are limit points of zeros of $T_n(z)$. For problem (ii) the author sketches a mode of attack. If $P_n(z) = z^n + \beta_1 z^{n-1} + \cdots + \beta_n$ is a given polynomial, then coefficients $\{a_m\}$ may be computed in terms of the $\{\lambda_m\}$ and $\{\beta_k\}$ so that

$$\sum_0^\infty \lambda_m z^{-m} = \sum_0^\infty a_m [P_n(z)]^{-m/n}.$$

Once the series $g(u) = \sum a_m u^m$ has been determined, find (1) the radius R_n of the largest circle within which $g(u)$ has no other singularities than algebraic branch-points, (2) for given $\{\lambda_m\}$ choose $P_n(z)$ so that R_n becomes a maximum. The maximum polynomial is $T_n(z)$.   *E. Hille.*

Hughes, H. K. On the asymptotic expansions of entire functions defined by Maclaurin series. Bull. Amer. Math. Soc. **50**, 425–430 (1944). [MF 10611]

Let (1) $f(z) = \sum_0^\infty g(n) z^n$ be entire, where $g(w)$ ($w = x + iy$, $g(w) = g(n)$ for $w = n$) satisfies the following conditions in the half-plane $x > x_0$ for arbitrary x_0: (i) $g(w)$ is single-valued and analytic; (ii) to every $\epsilon > 0$ corresponds a constant $K = K(\epsilon, x_0)$ such that (2) $|g(x+iy)| < Ke^{(\gamma+\epsilon)|y|}$, γ being a nonnegative constant. Newsom showed [Amer. J. Math. **60**, 561–572 (1938)] that

$$(3) \quad f(z) = \int_{-1-i}^{\infty} g(x)[(-1)^{k+1}z]^x \frac{\sin k\pi x}{\sin \pi x} dx$$
$$- \sum_{n=1}^{l} g(-n)z^{-n} + \xi(z, l),$$

where k is a positive integer with $k \geq \gamma$, and l is any positive integer, and where, if $|\arg [(-1)^{k+1}z]| < \pi$, then $z^l \xi(z, l) \to 0$ as $|z| \to \infty$. The present work, by further precising the function $g(w)$, replaces the integral in (3) so as to express $f(z)$ by means of a genuine asymptotic expansion. The principal theorem is as follows. Let the entire function $f(z)$ be given by (1), where $g(w)$ is single-valued and analytic in the (finite) w-plane, and where, in the half-plane $x > x_0$ (x_0 arbitrary) and for every $s = 0, 1, \cdots$,

$$(4) \quad g(w) = \sigma^w \left\{ \frac{c_0}{\Gamma(\alpha w + t)} + \frac{c_1}{\Gamma(\alpha w + t + 1)} + \cdots + \frac{c_s + \delta(\alpha w, s)}{\Gamma(\alpha w + t + s)} \right\},$$

with $\sigma, \alpha, t, c_0, \cdots, c_s$ constant (σ, α positive), and with $\delta(\alpha w, s) \to 0$ as $|w| \to \infty$, $s = 0, 1, \cdots$. Then, for large values of $|z|$ with $-\pi < \arg z \leq \pi$, $f(z)$ has the asymptotic development

$$(5) \quad f(z) \sim (1/\alpha) \sum_\mu \left\{ Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^\infty c_n Y_\mu^{-n} \right\} - \sum_{n=1}^\infty g(-n)z^{-n}.$$

Here $Y_\mu = \sigma^{1/\alpha} z^{1/\alpha} e^{2\pi i\mu/\alpha}$, and the summation $\sum_\mu$ is taken over all integers μ for which $|\arg z + 2\pi\mu| \leq \pi\alpha/2$.
                                    I. M. Sheffer (State College, Pa.).

Wolff, Julius. Inégalités remplies par les dérivées des fonctions holomorphes, univalentes et bornées dans un demi-plan. Comment. Math. Helv. **15**, 296–298 (1943).

The author gives a short proof, using a distortion theorem of Koebe and the Vitali covering theorem, of the following theorem of J. Ferrand [C. R. Acad. Sci. Paris **214**, 50–52 (1942); these Rev. **4**, 138]: if $f(z)$, $z = x + iy$, is analytic, bounded and univalent in $x > 0$, then, for almost all points it on the imaginary axis, $xf'(z) = O(|z - it|^{\frac{1}{2}})$ as $z \to it$. In the opposite direction, the author constructs for each positive ϵ a function $f(z)$, analytic, bounded and univalent in $x > 0$, such that, for points it in a residual set on the imaginary axis, $\lim \sup x^{1-\epsilon}|f'(z)| = \infty$ as $z \to it$ on any curve $|y-t| = x^p$, $p > 0$.   *R. P. Boas, Jr.* (Cambridge, Mass.).

Speiser, Andreas. Über symmetrische analytische Funktionen. Comment. Math. Helv. **16**, 105–114 (1944).

The structure of the group of mappings of a simply connected Riemann surface onto itself is used to determine and discuss the function having the Riemann surface. The particular cases considered lead to certain modular functions and elliptic integrals.   *H. S. Zuckerman* (Seattle, Wash.).

Banin, A. M. Approximate conformal transformation applied to a plane parallel flow past an arbitrary shape. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 131–140 (1943). (Russian. English summary) [MF 9732]

The problem of conformal mapping of the exterior of a contour L onto the exterior of the unit circle can be reduced to an integral equation. Gershgorin [Rec. Math. [Mat. Sbornik] **40**, 48–58 (1933)] derived this equation in the form

$$\theta(s) = -\pi^{-1} \int_L [\theta(\sigma) \cos (n_\sigma, r)/r] d\sigma + 2[\phi(s) + \pi].$$

Here $\exp[i\theta(\sigma)]$ denotes the image on the unit circle of that point P_σ of L which is at a distance σ from a fixed point P_0, the distance being measured along L; r is the length of the straight line connecting P_σ and P_s, (n_σ, r) the angle between the normal at P_σ and the line $\overline{P_\sigma P_s}$, and $\phi(s)$ the angle between $\overline{P_s P_0}$ and the positive x-axis. The author transforms the Gershgorin equation into the form

$$(*) \quad \theta'(s) = \pi^{-1} \int_L \theta'(\sigma)[(\partial\alpha(s, \sigma)/\partial s) - a(s)] d\sigma + 2a(s),$$

where $\alpha(s, \sigma)$ is the angle between $\overline{P_s P_\sigma}$ and the positive x-axis and $a(s)$ an arbitrary continuous function of s. He then replaces (*) by a (finite) system of linear difference equations. The method described above is illustrated by an example.   *S. Bergman* (Providence. R. I.).

Smith, C. A. B. On the definitions of elliptic functions. Math. Gaz. **28**, 41–45 (1944). [MF 10625]

The important step in setting up elliptic functions as inverses of elliptic integrals is to show that they are meromorphic over the finite plane. A fairly direct and simple proof of this fact is given making use of the symmetry properties of the elliptic functions. The proof is not as short as one involving the addition theorem [Goursat, Cours d'analyse mathématique] but it may have the advantage of seeming less artificial.   *H. S. Zuckerman.*

Neville, E. H. Jacobian elliptic functions. J. London Math. Soc. **18**, 177–191 (1943). [MF 10370]

This is the text of a lecture in which the author makes a plea for the revival of the study of the Jacobian elliptic functions. He criticizes the usual treatment and suggests a more suitable approach. Of special interest is a notational

device which, while neither awkward nor radically different from that usually used, is a considerable aid to the memory.

H. S. Zuckerman (Seattle, Wash.).

van Veen, S. C. Die Berechnung der vollständigen elliptischen Integrale erster und zweiter Art für grosse Werte von |k|. Nederl. Akad. Wetensch., Proc. 45, 171–175 (1942). [MF 10385]

The author gives a new proof of two relations which express $K(k)$ and $E(k)$ in terms of K and E for arguments $1/k$ and $(1-k^{-2})^{\frac{1}{2}}$ [Hermite, Cours, Hermann, Paris, 1883, pp. 191–196]. *P. Franklin* (Cambridge, Mass.).

van Veen, S. C. Stark konvergente Entwicklungen für die Funktionen $D(k)$ und $C(k)$. Nederl. Akad. Wetensch., Proc. 45, 240–248 (1942). [MF 10391]

The calculation of the induction of circular coils leads to two functions $D(k)$ and $C(k)$, where $k^2D=K-E$ and $k^2C=2D-K$ express these functions in terms of complete elliptic integrals. The author uses Landen's transformation to derive new series which converge rapidly in certain regions for k which had previously proved troublesome. He also obtains approximate formulas and investigates their error. *P. Franklin* (Cambridge, Mass.).

Bergman, Stefan and Schiffer, Menahem. Bounded functions of two complex variables. Amer. J. Math. 66, 161–169 (1944). [MF 10566]

Preparatory to the main theorem the authors prove the following lemma. Lemma 1. Let $\{F_n(Z, \lambda)\}$ $(n=1, 2, \cdots)$ be a set of functions defined in $U=E[|Z|<1, 0\le\lambda\le2\pi]$, which are uniformly bounded; that is, let there exist a constant A, such that $|F_n(Z, \lambda)|\le A$, $(Z, \lambda)\epsilon U$. For every fixed λ, let the $F_n(Z, \lambda)$ be analytic functions of Z, $|Z|<1$. Finally let them be uniformly continuous in λ in the following sense: for every $r<1$ and $\epsilon>0$ there exists a $\delta(\epsilon, r)\to0$ as $\epsilon\to0$ such that

$$(1) \qquad |F_n(Z, \lambda_1)-F_n(Z, \lambda_2)|\le\epsilon$$

for $|\lambda_1-\lambda_2|\le\delta(\epsilon, r)$ and $|Z|\le r<1$. Under these assumptions the $F_n(Z, \lambda)$ form a normal family in U. Next the authors consider a segment of an analytic hypersurface which is a 3-dimensional mainfold with the parametric representation (2) $z_j=h_j(Z, \lambda)$, $j=1, 2$, where the h_j are functions of Z and λ, defined for $[0\le\lambda\le2\pi, Z\epsilon B(\lambda)]$, having continuous derivatives with respect to λ and Z and analytic in Z for Z in $B(\lambda)$ for each fixed λ. Certain additional hypotheses are placed on the functions h_j. Denoting by I the 3-dimensional hypersurface in the 4-dimensional (z_1, z_2)-space which is obtained by the mapping (2), the authors divide the points of I into two classes: J-points and K-points. A point of I is a J-point if it corresponds to an interior point of $B(\lambda)$ and if λ does not belong to a certain denumerable set of exceptional values.

The main theorem is as follows. Let M be a 4-dimensional domain in (z_1, z_2)-space whose boundary contains the segment I of an analytic hypersurface. Let $\{f_n(z_1, z_2)\}$ be a family of functions which are analytic and uniformly bounded in $M+I$. If in addition the functions $f_n(z_1, z_2)$ are such that the functions $F_n(Z, \lambda)=f_n[h_1(Z, \lambda), h_2(Z, \lambda)]$ satisfy condition (1), then the sequence $f_n(z_1, z_2)$ forms a normal family in $M+I$ and, in each J-point (t_1, t_2) of I,

$$\lim_{n\to\infty}f_n(t_1, t_2)=f(t_1, t_2)=\lim_{(z_1, z_2)\to(t_1, t_2)}f(z_1, z_2)$$

holds for each convergent partial sequence of f_n. The proof

is based upon lemma 1, a second lemma which concerns a family of harmonic functions depending upon a real parameter and a third lemma which applies lemmas 1 and 2, to the set of analytic functions $\{f_n(z_1, z_2)\}$. The proof also uses certain results previously obtained by Bergman.

W. T. Martin (Syracuse, N. Y.).

Lelong, Pierre. Sur les valeurs lacunaires d'une relation à deux variables. Bull. Sci. Math. (2) 66, 103–108, 112–125 (1942). [MF 10468]

Let $F(x, y)$ be an analytic function of two complex variables in a product domain $[x\epsilon d, |y|<\infty]$, where d is domain in the x-plane. A value y_0 is called a lacunary value of the relation (1) $F(x, y)=0$ if the relation $F(x, y_0)=0$ has no solution for x in d. Denote by $E(y)$ the set of lacunary values of the relation (1). In 1926 Julia [Bull. Soc. Math. France 54, 26–37 (1926)] posed the question of determining the power of the set $E(y)$ in the case in which $f(x, y)$ is an entire function. Two particular results when F is entire are: (1) if F is linear in y, the relation (1) becomes $y-f(x)=0$, $f(x)$ being entire, and by Picard's theorem $E(y)$ contains at most one point; and (2) if $F(x, y)$ is a polynomial of degree p in y, the set $E(y)$ consists of at most $2p-1$ points in the finite plane. In the paper being reviewed the author establishes the denumerable character of the set $E(y)$ in a third case, namely, that in which F is a function whose maximum growth is of finite order in x, in a sense to be specified. The author handles both the case in which F is entire as well as the case in which F is merely analytic in a domain $[x\epsilon d, |y|<\infty]$.

Let g be any closed domain interior to the domain d, and denote by $M(r, g)$ the maximum of $|F(x, y)|$ for $|x|\le r$, $y\epsilon d$. If the limit superior

$$\alpha(g)=\lim\sup_{r\to\infty}[\log_2 M(r, g)/\log r]$$

is finite for every such domain g, then F is said to be of finite order in x. (In general $\alpha(g)$ depends upon the domain g. If, however, d is the entire y-plane so that F is an entire function, then the author shows that $\alpha(g)$ is independent of g. The proof of this fact uses the convexity property of the function $M(r, r')=\max_{|x|\le r, |y|\le r'}|F(x, y)|$.) By use of the Hartogs series development for $F(x, y)$ the author proves the following theorem. If $F(x, y)$ is analytic in $[x\epsilon d, |y|<\infty]$ and of finite order in x then the set $E(y)$ has no limit points in d; it is a denumerable set whose derived set is located on the boundary of d. As a corollary to this result he obtains the result that, if F is an entire function of finite order, then there are at most a finite number of lacunary values of $F=0$ in any bounded portion of the y-plane. Another result relates the behavior of the number of points of the set $E(y)$ in the circle $|y|\le r'$ as $r'\to\infty$ to the behavior of the function $M(1, r')$. The author also studies the relation (1) in the neighborhood of a lacunary value y_0 and shows, for example, that in such a neighborhood there are an infinite number of bicylinders $|y-y_0|<r'$, $|x|<r$ into which the variety defined by $F(x, y)=0$ can not penetrate.

W. T. Martin (Syracuse, N. Y.).

Theory of Series

Buck, R. Creighton. Limit points of subsequences. Bull. Amer. Math. Soc. 50, 395–397 (1944). [MF 10605]

With general topological criteria for convergence, it is shown that divergence of a multiple sequence implies divergence of almost all subsequences. *R. P. Agnew.*

Day, M. M. Cluster points of subsequences. Bull. Amer. Math. Soc. **50**, 398–404 (1944). [MF 10606]

Generalizations of the preceding result of Buck, involving more general concepts of limit points and subsequences.

R. P. Agnew (Ithaca, N. Y.).

Erdös, P. A note on Farey series. Quart. J. Math., Oxford Ser. **14**, 82–85 (1943). [MF 9934]

The author brings certain earlier results of A. E. Mayer [Quart. J. Math., Oxford Ser. **13**, 185–192 (1942); these Rev. **4**, 194] to the following conclusion. There is an absolute constant $c>0$ such that, if $n>ck$ and if a_1/b_1, a_2/b_2, $\cdots$ denotes the Farey sequence of order n, the fractions a_x/b_x and a_{x+k}/b_{x+k} are similarly ordered.

G. Szegö (Stanford University, Calif.).

Hill, J. D. Some properties of summability. II. Bull. Amer. Math. Soc. **50**, 227–230 (1944). [MF 10202]

[The first part appeared in Duke Math. J. **9**, 373–381 (1942); these Rev. **3**, 295.] Let A be a regular matrix method of summability and let B be the familiar space of bounded sequences. It is shown by use of the diagonal process that, if S is a separable subset of B, then there is a method A', whose matrix consists of some of the rows of the matrix of A, such that each sequence in S is summable A'. If A is reversible (such that the equations $\sum a_{nk} s_k = t_n$ have a unique solution corresponding to each convergent sequence t_n) and of type M (such that $\sum |u_n| < \infty$ and $\sum a_{nk} u_k = 0$ imply $u_n = 0$) and if the convergence field of A contains at least one divergent sequence, then the convergence field of A contains some unbounded sequences. If A is reversible and not of type M, its convergence field contains some unbounded sequences. *R. P. Agnew*.

Knopp, Konrad. Über eine Erweiterung des Äquivalenz-satzes der C- und H-Verfahren und eine Klasse regulär wachsender Funktionen. Math. Z. **49**, 219–255 (1943). [MF 10036]

Let S denote the set of complex-valued functions $s(t)$, defined for $t>0$, which are bounded and integrable (Lebesgue) over $0 \leq t \leq a$ for each $a>0$. For each $s \varepsilon S$ and complex constant k with real part positive, let

$$C_k(x; s(t)) = (k/x) \int_0^x (1-t/x)^{k-1} s(t) dt,$$

$$H_k(x, s(t)) = (\Gamma(k))^{-1} x^{-1} \int_0^x (\log(x/t))^{k-1} s(t) dt$$

denote the Cesàro and Hölder transforms of $s(t)$. Let P_a denote the set of complex-valued continuous "regulär wachsend" functions $p(x)$ for which

$$\frac{1}{x|p(x)|} \int_0^x |p(t)| dt \leq M, \quad \lim_{x \to \infty} \frac{1}{xp(x)} \int_0^x p(t) dt = a,$$

the first inequality holding for all $x>0$. If k is a positive integer, $s \varepsilon S$, $p \varepsilon P_a$, and if one of the two limits

$$(1) \qquad \lim_{x \to \infty} \frac{C_k(x, s(t))}{p(x)} = g_k, \quad \lim_{x \to \infty} \frac{H_k(x, s(t))}{p(x)} = h_k$$

exists, then both exist and

$$h_k = [(1+a)(1+2a) \cdots (1+\overline{k-1}a)/k!] g_k.$$

If $k>0$ (not necessarily an integer), $s \varepsilon S$, $p \varepsilon P_a$, where $a \neq 0$, and if one of the two limits (1) exists, then both exist and

$$h_k = [\Gamma(k+a^{-1})/\Gamma(k+1)\Gamma(a^{-1})] a^k g_k.$$

In case k is positive but not an integer, it is not known whether the hypothesis that $p \varepsilon P_0$ implies that, for each $s \varepsilon S$, the limits in (1) both exist or both fail to exist. For each $s \varepsilon S$, let

$$A(x, s(t)) = x^{-1} \int_0^\infty e^{-t/x} s(t) dt$$

denote the Abel-Laplace transform of $s(x)$. If $k>0$, $s \varepsilon S$, $p \varepsilon P_a$, where $a \neq 0$, and if the limits in (1) exist, then

$$\lim_{x \to \infty} \frac{A(x, s(t))}{p(x)} = \frac{\Gamma(a^{-1})}{a^k} g_k.$$

The proofs are based on several lemmas concerning functions in the sets P_a and on relations between Cesàro, Hölder and Abel-Laplace transformations previously obtained [same Z. **47**, 229–264 (1941); these Rev. **3**, 296] by the author. *R. P. Agnew* (Ithaca, N. Y.).

Hadwiger, H. Über ein Distanz-theorem bei der A-Limitierung. Comment. Math. Helv. **16**, 209–214 (1944).

It is shown that the following assertion is true when $\rho = 1.0160 \cdots$ and false when $\rho = .4858 \cdots$. Let $c_0 + c_1 + \cdots$ be a series of complex constants for which $\limsup n|c_n| < \infty$. Let L denote the set of limit points of the sequence of partial sums of $\sum c_n$. Let L_A denote the set of limit points of the Abel transform $\sigma(t) = \sum_{n=0}^\infty c_n t^n$ $(0<t<1)$ of $\sum c_n$; $y \varepsilon L_A$ if there is a sequence t_k such that $0<t_k<1$, $t_k \to 1$, and $\sigma(t_k) \to y$. Then to each $x \varepsilon L$ corresponds a $y \varepsilon L_A$, and to each $y \varepsilon L_A$ corresponds an $x \varepsilon L$, such that $|x-y| \leq \rho \limsup n|c_n|$. The hypothesis that $nc_n \to 0$ implies that L and L_A are identical; the weaker hypothesis $n|c_n| < K$ does not.

R. P. Agnew (Ithaca, N. Y.).

Silverman, L. L. and Szász, O. On a class of Nörlund matrices. Ann. of Math. (2) **45**, 347–357 (1944). [MF 10271]

Let N_p denote the Nörlund sequence-to-sequence transformation $y_n = P_n^{-1} \sum_{r=0}^n p_{n-r} x_r$, determined by a sequence p_0, p_1, $\cdots$ for which $P_n \equiv p_0 + \cdots + p_n \neq 0$, $n = 0, 1, 2, \cdots$. For each $k = 1, 2, \cdots$, let Z_k denote the transformation for which $p_j = 1$ when $0 \leq j < k$ and $p_j = 0$ when $j \geq k$. After a general discussion of the methods N_p, the transformations Z_k and their powers are related to each other and to the arithmetic mean transformation. One result is the following. If k is a multiple of h, then Z_h includes Z_k; if k and h are relatively prime, then the only sequences summable both Z_h and Z_k are convergent. A criterion is given for equivalence of I (convergence) and $\alpha I + (1-\alpha) Z_k$; this problem was solved by Kubota [Tôhoku Math. J. **12**, 222–224 (1917)] for the case in which Z_k is any Nörlund method for which $p_j = 0$ when $j \geq k$. If $\sum u_n$ and $\sum v_n$ are summable N_p and N_q, respectively, if $p_n \geq 0$ and $q_n \geq 0$, and if one of the summabilities is absolute, then the Cauchy product series is summable by the "symetric product" method N_r, generated by the sequence $r_n = \sum p_{n-k} q_k$. *R. P. Agnew*.

Sheffer, I. M. Systems of linear equations of analytic type. Duke Math. J. **11**, 167–180 (1944). [MF 10157]

Let the type of a sequence $\{y_n\}$ be defined as

$$\limsup_{n \to \infty} |y_n|^{1/n}.$$

The author discusses conditions under which the transformation $\sum_{k=0}^\infty a_{nk} x_k = c_n$ carries a sequence $\{x_k\}$ of one class into a sequence $\{c_n\}$ of another class, the classes considered being those of type $\leq r$, $<r$, $<\infty$ or $\leq \infty$; he gives characterizations of the transformation in each of the 16

possible cases. Let $A_n^*(t) = \sum_{k=0}^{\infty} |a_{nk}| t^k$, with radius of convergence r_n, and let $r^* = \inf r_n$. Specimen result: the transformation carries the class of sequences of type $\leq r$ into that of sequences of type $\leq h$ if and only if $r^* > r$ and

$$\lim_{r' \to r+} \lim \sup_{n \to \infty} A_n^*(r') \leq h.$$

Various properties of $\lim \sup A_n^*(r)$ and of related numbers are discussed. A final section deals with the product of two transformations. *R. P. Boas, Jr.* (Cambridge, Mass.).

Fourier Series and Generalizations, Integral Transforms

Wang, Fu Traing. On strong summability of a Fourier series. Bull. Amer. Math. Soc. **50**, 412–416 (1944). [MF 10608]

Let $s_n(x)$ be the nth partial sum of the Fourier series of $f(x)$ and let

$$\phi(t) = \{f(x+t) + f(x-t) - 2f(x)\}/2.$$

If

$$\int_0^t |\phi(u)| \{1 + \log^+ |\phi(u)|\} du = o(t),$$

then

$$\sum_0^n |s_r(x) - f(x)|^2 = o(n \log \log n)$$

as $n \to \infty$. *A. Zygmund* (South Hadley, Mass.).

Wang, Fu Traing. On the summability of Fourier series by Riesz's typical means. J. London Math. Soc. **18**, 155–160 (1943). [MF 10367]

Let $(\dagger)$ $\phi(t) \sim \sum_1^\infty a_n \cos nt$. It has been proved that, if

$$\int_0^t \phi(u) du = o(t/\log(1/t))$$

as $t \to 0$, the Fourier series $(\dagger)$ is summable by the Riesz means $(e^{(\log n)^2}, 2)$ to sum 0 at $t = 0$. The object of the paper is to show that under the same conditions the series is summable $(e^{(\log n)^2}, 1 + \delta)$ to sum 0 at $t = 0$ for any $\delta > 0$ and that the result is false for $\delta = 0$. *R. Salem.*

Wang, Fu Traing. A note on Riesz summability of the type e^{n^a}. Bull. Amer. Math. Soc. **50**, 417–419 (1944). [MF 10609]

Let σ_n^2 denote the rth Cesàro means $(r = 1, 2, \cdots)$ of the series $\sum a_n$. (i) If $\sigma_n^2 - s = o(n^{-r\alpha})$, $0 < \alpha < 1$, as $n \to \infty$, then the series $\sum a_n$ is summable (e^{n^α}, τ) to the sum s, where $\tau > r/(1-\alpha)$. As an application the following result is obtained. (ii) If $\sigma_n^2 - s = o(n^{-r\alpha})$ and if $a_n > -Kn^{-1}$, the series $\sum a_n$ converges to sum s. *A. Zygmund.*

Wang, Fu Traing. On Riesz summability of Fourier series by exponential means. Bull. Amer. Math. Soc. **50**, 420–424 (1944). [MF 10610]

Let $f(x)$ be of period 2π and let

$$\phi(t) = \{f(x+t) + f(x-t) - 2f(x)\}/2,$$

$$-\phi_\beta(t) = (1/\Gamma(\beta)) \int_0^t (t-u)^{\beta-1} \phi(u) du.$$

Let a_n, b_n be the Fourier coefficients of f. If

$$a_n \cos nx + b_n \sin nx > -Kn^{-\beta/\gamma}, \qquad 0 < \beta < \gamma,$$

and if $\phi_\beta(t) = o(t^\gamma)$ as $t \to 0$, then the Fourier series of f converges at the point x to sum $f(x)$. *A. Zygmund.*

Kuttner, B. Note on the Riesz means of a Fourier series. J. London Math. Soc. **18**, 148–154 (1943). [MF 10366]

It is well known that, if $f(x)$ is everywhere positive, the Cesàro means of order 1 of the Fourier series of $f(x)$ are also everywhere positive. The author shows that this result holds for the Riesz means (R, n, k) of type n and of any order $k \geq 1$; it holds also for the Riesz means (R, n^λ, k) when $k \geq 1$, provided that $\lambda < 1$. However, the Riesz means of the type n^λ, (R, n^λ, k), fail to have an analogous property when $\lambda \geq 2$, however large k may be. Furthermore, a corresponding result holds for the Abel means of the type n^λ, (A, n^λ), for $\lambda > 2$ but not for $\lambda = 2$. The case $1 < \lambda < 2$ was left open but a note added in proof indicates that the author has proved that the means (A, n^λ) are positive for $\lambda \leq 2$, also that if $1 < \lambda < 2$ the means (R, n^λ, k) are all everywhere positive provided that k is taken large enough. The proofs of the last results will be published later. *R. Salem* (Cambridge, Mass.).

Hille, Einar. On the oscillation of differential transforms and the characteristic series of boundary-value problems. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 161–168 (1944). [MF 10460]

An expository talk. The author discusses the Pólya-Wiener results on the zeros of successive derivatives of periodic functions and his own extensions to the case of differential operators. *A. Zygmund.*

Pollard, Harry. A new criterion for completely monotonic functions. Trans. Amer. Math. Soc. **55**, 457–464 (1944). [MF 10503]

A function $f(x)$ is completely monotonic in $0 \leq x < \infty$ if it has derivatives of all orders and $(-1)^n f^{(n)}(x) \geq 0$ there $(n = 0, 1, 2, \cdots)$; or [S. Bernstein] if $(*)$ $(-1)^n \Delta_h^n f(x) \geq 0$ $(x \geq 0; n = 0, 1, 2, \cdots; h > 0)$; or [Bernstein, Widder] if $(\dagger)$ $f(x) = \int_0^\infty e^{-xt} dF(t)$, $0 \leq x < \infty$, $F(t)$ nondecreasing and bounded. The author replaces $(*)$ by a condition in which the sign of the nth difference is prescribed only for one value of h for each n, instead of for all positive h; his criterion is that $f(x)$ is continuous in $0 \leq x < \infty$, $f(\infty)$ exists and $(-1)^n \Delta_{h_n}^n f(x) \geq 0$ for an infinite sequence of integers n, where $h_n > 0$, $h_n = o(1/n^2)$ as $n \to \infty$. The proof proceeds by showing that this criterion implies the representation $(\dagger)$ for $f(x)$. This follows by standard methods once we have, when $f(\infty) = 0$,

$$\lim_{n \to \infty} \frac{(-1)^n}{h_n{}^n(n-1)!} \int_0^\infty e^{-nx/u} u^{n-1} \Delta_{h_n}^n f(u) du = f(x),$$

and this relation is derived from a lemma, analogous to repeated integration by parts, which effectively removes the difference operators from $f(u)$ and makes them apply to the rest of the integrand. *R. P. Boas, Jr.*

Schelkunoff, S. A. Proposed symbols for the modified cosine and exponential integrals. Quart. Appl. Math. **2**, 90 (1944). [MF 10340]

Lieblein, Viktor. Über einen vierfachen Integrator. Monatsh. Math. Phys. **50**, 128–141 (1941). [MF 10477]

If the Fredholm operator $\int_a^b K(x, u)\phi(u) du$ transforms a continuous function $\phi(x)$ into $\Phi(x)$, the real and continuous kernel $K(x, u)$ associated with this operator is said to be a r-tuple integrator if $d^r \Phi(x)/dx^r = \phi(x)$ identically in ϕ and $\Phi(x)$ satisfies certain boundary conditions. In this paper the

most general integrator of the fourth order is found for which $\Phi^{(iv)}(x)=\phi(x)$ and $\Phi(a)=\Phi(b)=\Phi'(a)=\Phi'(b)=0$. It is shown that this integrator is symmetric and that its characteristic values are positive. Explicit expressions for the integrator, its characteristic values and its characteristic functions are given. A certain quadrature formula is deduced which is a refinement of the trapezoidal rule.

> I. A. Barnett (Cincinnati, Ohio).

Special Functions

Colombo, 🔲. S. Sur quelques correspondances symboliques. C. R. Acad. Sci. Paris 216, 368–369 (1943). [MF 10018]

The author discusses the properties of the function

$$v(x)=\int_0^\infty \frac{x^t dt}{\Gamma(t+1)}$$

by means of the operational calculus. From the operational representation $1/\log p$ of $v(x)$, the integral equation

$$2(\pi x)^{\frac12}v(x)=\int_0^\infty v(t)\exp(-t^2/4x)dt$$

and other properties of $v(x)$ are deduced. The function

$$v_m(x)=\int_0^\infty \frac{x^t t^m dt}{\Gamma(t+1)}$$

has the operational representation $\Gamma(m+1)/\log^{m+1}p$ and can be studied similarly.   A. Erdélyi (Edinburgh).

Banerjee, D. P. On some infinite integrals. Proc. Benares Math. Soc. (N.S.) 4, 1–2 (1943). [MF 10341]

Certain infinite integrals involving the function

$$|\Gamma(a+ix)/\Gamma(b+ix)|^2$$

are evaluated by use of an identity due to Ramanujan.

> H. Pollard (New York, N. Y.).

Shabde, N. G. On some integrals involving Legendre functions. Proc. Benares Math. Soc. (N.S.) 4, 3–8 (1943). [MF 10342]

The integral

$$\int_{-1}^1 P_l^{m'}(z)P_n^m(z)dz/(1-z^2), \qquad m-m'=2k,$$

is expressed as a series when $l=n-2r$, k, n and r being positive integers. It is shown to be zero when n is odd and l even. This is also the case when n is given and $l\geq n$. By means of this result and a known expansion it is shown that

$$\int_{-1}^1 (1-z^2)^{\frac12 m-1}P_p^m(z)P_q^m(z)P_n^{m'}(z)=0$$

(1) when $m-m'=2k$, $n\geq p+q+m$; (2) $m-m'=2k$ and n odd, $p+q+m$ even or n even, $p+q+m$ odd. In other cases the value is complicated. Two integrals involving the product $P_p^m(\operatorname{ch}u)Q_q^m(\operatorname{ch}u)$ are evaluated for the range $u=0$ to ∞ and a simple value is found for the integral

$$\int_{-1}^1 P_n(t)f_n(t)dt,$$

where $f_n(t)$ is a certain polynomial of degree n.

> H. Bateman (Pasadena, Calif.).

Schwarz, L. Untersuchung einiger mit den Zylinderfunktionen nullter Ordnung verwandter Funktionen. Luftfahrtforschung 20, 341–372 (1944). [MF 10579]

The author discusses the properties of the four integrals

$$\int_0^\infty J_0(\lambda u)\frac{\cos u}{\sin u}du,\quad \int_0^\infty N_0(\lambda u)\frac{\cos u}{\sin u}du$$

and some similar integrals which occur in aerodynamic theory. Convergent and asymptotic series expansions are obtained and relations among the integrals for special values of the parameters λ and x. Finally the author tabulates numerical values of the integrals to six decimal places in the ranges: λ, 0(.1)1; x, 0(.02)2, 2(.1)5.

> M. C. Gray (New York, N. Y.).

Rutgers, J. G. On series and definite integrals involving Bessel functions. I. Nederl. Akad. Wetensch., Proc. 45, 376–379 (1942). (Dutch) [MF 10406]

Rutgers, J. G. On series and definite integrals involving Bessel functions. II. Nederl. Akad. Wetensch., Proc. 45, 484–489 (1942). (Dutch) [MF 10414]

Sinha, S. Some infinite integrals involving Bessel functions of imaginary argument. Bull. Calcutta Math. Soc. 35, 37–42 (1943). [MF 9947]

Evaluation of the integral

$$\int_0^\infty x^{\rho-1}K_\lambda(ax)K_\mu(ax)\,_pF_q(bx^2)dx$$

by expanding the generalized hypergeometric function $_pF_q$ and integrating term-by-term; the result is expressed in terms of $_{p+4}F_{q+2}(b/a^2)$. The author's statement that term-by-term integration is justifiable if $q-p\geq 1$ needs modification, for, if $q=p+1$, the resulting series is not convergent unless $|b|<|a|^2$. A great number of particular cases are written out explicitly; in these cases $_pF_q$ reduces to a Bessel function, a product of Bessel functions, a product of confluent hypergeometric functions, of Laguerre polynomials. Other particular cases are only mentioned.

> A. Erdélyi (Edinburgh).

Gupta, H. C. Some infinite integrals. Proc. Benares Math. Soc. (N.S.) 4, 45–50 (1943). [MF 10349]

Some infinite integrals involving the product of one Whittaker function and one or more Bessel functions or hypergeometric functions are evaluated. The method used is essentially an inversion of the Laplace integral, applied to known identities.   H. Pollard (New York, N. Y.).

Gupta, H. C. Some self-reciprocal functions. Bull. Calcutta Math. Soc. 35, 67–70 (1943). [MF 9951]

A function $f(x)$ is said to be R_ν if

$$f(x)=\int_0^\infty (xy)^{\frac12}J_\nu(xy)f(y)dy.$$

Employing three known transformations of R_ν functions into $R_{3\nu-1}$, $R_{3\nu-1}$ and $R_{1-\nu}$ functions, respectively, and applying these transformations to the well-known $R_{\nu+2m}$ function

$$x^{\nu+2m+\frac12}e^{-\frac12 x^2}L_\nu^m(\tfrac12 x^2),$$

four R functions are obtained, all of them of the form $x^{\alpha-1}\,_3F_3(-\frac12 x^2)$, where α and the parameters of the generalized hypergeometric function $_3F_3$ have appropriate values, for which we refer to the original paper.   A. Erdélyi.

Mital, P. C. On self-reciprocal functions. Proc. Benares Math. Soc. (N.S.) **4**, 41–43 (1943). [MF 10348]

A function is said to be R_s or R_c if it is respectively its own sine or cosine transform. It is well known that, at least formally, the Laplace transform of an R_s is R_c, and vice versa. In an earlier note [J. Indian Math. Soc. (N.S.) **6**, 25–32 (1942); these Rev. **4**, 99] the author has evaluated the Laplace transform of a certain product of Bessel functions; he now observes that it furnishes R_s and R_c functions in the form of sums of hypergeometric functions of the types $_2F_2$ and $_3F_4$. Using another transformation, which for R_s functions is formally equivalent to the Laplace transform, he writes down some R_c functions as definite integrals; the values which he gives for the definite integrals appear to the reviewer to be incorrect. *R. P. Boas, Jr.*

Mitra, S. C. On certain transformations in generalized hypergeometric series. J. Indian Math. Soc. (N.S.) **7**, 102–109 (1943). [MF 10107]

Some rather complicated formulas are obtained by formal manipulation of well-known results for generalized hypergeometric series with argument ± 1. The simplest, and probably most interesting, theorem is that any hypergeometric function of the form

$$F(a, b; \tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2}e+\tfrac{1}{2}; \tfrac{1}{2})$$

can be expressed as a finite series whenever e is an integer. For example, when $e=1$ the function has the value

$$2\pi^{\frac{1}{2}}\frac{\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}b+1)}{b-a}\left[\frac{1}{\Gamma(\tfrac{1}{2}b)\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})}-\frac{1}{\Gamma(\tfrac{1}{2}a)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2})}\right].$$

M. C. Gray (New York, N. Y.).

Basu, K. A note on the oscillating rotator. Indian J. Phys. **17**, 193–196 (1943). [MF 9979]

The wave equation for the oscillating rotator as given by Fues and Sommerfeld is

$$F''-m(m+1)F/\rho^2+8\pi^2h^{-2}J(E-U)F=0,$$

where the potential energy U has the form due to Kratzer:

$$U=A-J\omega_0^2(1/\rho-1/2\rho^2+b(\rho-1)^3+c(\rho-1)^4+\cdots).$$

The author treats this equation in the usual manner, taking $b=c=0$ as a first approximation and then using perturbation methods. By applying to the unperturbed equation a different transformation than that previously used by Fues and Chakravarti, the author obtains the equation: $su''+(2\gamma-s)u'+nu=0$ for Sonine polynomials recently studied by him [Bull. Calcutta Math. Soc. **35**, 21–32 (1943); these Rev. **5**, 180]. He then expands the perturbed eigenfunction in a series of Sonine polynomials. Using relations which he has previously derived for these polynomials, he finds explicit formulas for the perturbed eigenvalue in terms of γ, n and the constants b and c which occur in the potential energy function. This treatment is rather simpler than that of Chakravarti, who expanded the eigenfunction in terms of Hermite polynomials rather than Sonine polynomials. *O. Frink* (State College, Pa.).

Differential Equations

Sispánov, Sergio. On a differential equation of second order. Revista Union Mat. Argentina **9**, 165–170 (1943). (Spanish) [MF 10125]

The equation treated is

$$\left(\log\frac{1+y'}{y'}\right)'=x^{-1}+y^{-1}-(x+y)^{-1}.$$

Valentine, F. A. On the convergence of an iteration process for the differential equation $dx/dt=f(t, x, x')$. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 77–84 (1944). [MF 10456]

The paper shows that a single application of the Picard iteration process makes possible an existence theorem for a differential equation in the form $x'=f(t, x, x')$, where t ranges over the real interval T, x is on T to B, a complete normed linear space, x' is dx/dt and f is on the product space TBB to B. It is assumed that f is continuous and bounded on R: $|t-t_0|\leq A$, $\|x-x_0\|\leq C$ and $\|x'\| < \infty$ and satisfies a Lipschitz condition

$$\|f(t, x_2, x_2')-f(t, x_1, x_1')\|\leq L\|x_2-x_1\|+K\|x_2'-x_1'\|,$$

with $L<0$ and $0<K<1$. Uniqueness of the solution in a suitably defined neighborhood of t_0, which satisfies the initial condition $x_0'=f(t_0, x_0, x_0')$, is obtained. Modification of the region R and changing the form of the Lipschitz condition give rise to variants of the principal result.

T. H. Hildebrandt (Ann Arbor, Mich.).

Fleckenstein, J. O. Über eine verallgemeinerte Hillsche Determinante. Comment. Math. Helv. **15**, 367–376 (1943).

Application of Hill's methods to the system of differential equations

$$y_s''(t)+\sum_{k=1}^{n} \theta_{sk}(t)y_k(t)=0, \quad s=1, 2, \cdots, n,$$

in which the coefficients $\theta_{sk}(t)$ are given functions, with period 2π, having Fourier expansions

$$\theta_{sk}(t)= \sum_{m=-\infty}^{\infty} \theta_{skm}e^{imt}$$

such that $\sum_{m=-\infty}^{\infty}|\theta_{skm}| < \infty$ and the n numbers θ_{ss0} are distinct. *R. P. Agnew* (Ithaca, N. Y.).

Bitterlich-Willmann, Johann. Über die Asymptoten der Lösungen einer Differentialgleichung. Monatsh. Math. Phys. **50**, 35–39 (1941). [MF 10490]

For the equation (*) $y''+F(x)y=0$ with $F(x)$ continuous, the author proves the following. (1) If, for $x\geq c>0$, $x^{2+\alpha}|F(x)| <C$, then each solution of (*) has an asymptote (α and C being positive constants). (2) If $F(x)$ has a constant sign for $x\geq c>0$ and also

$$C_1/x^2 < |F(x)| < C_2/x^{2+\beta},$$

C_1, C_2, β positive constants, then, if any solution of (*) has an asymptote, it must be parallel to the x-axis. (3) If $F(x)$ has a constant sign for $x\geq c>0$ and if $C<x^2|F(x)|$, then, if any solution of (*) has an asymptote, it must be the x-axis. [For a generalization of these results see a paper by Haupt, Math. Z. **48**, 212–220 (1942); these Rev. **4**, 276.]

F. G. Dressel (Durham, N. C.).

van der Corput, J. G. On the uniqueness of solutions of differential equations. Nederl. Akad. Wetensch., Proc. **45**, 136–138 (1942). [MF 10380]

Let

$$p(x)=\sum_{\nu=0}^{n} \eta_{\nu+1}((x-\xi)^\nu/\nu!),$$

where ξ and the η_i are real numbers, and let

$$f(x; y_1, \cdots, y_n)=f(x, y)$$

be defined in $\xi<x<\xi+w$, $-\infty <y_i<\infty$ $(w>0; i=1, \cdots, n)$.

Suppose, furthermore, that, for any two points $(x, y_1, \cdots, y_n)$ and $(x, Y_1, \cdots, Y_n)$ of the region

$$\xi < x < \xi + w, \quad |y_i - d^i p(x)/dx^i| < w((x-\xi)^{n-i}/(n-i)!),$$

$f(x, y)$ satisfies the inequality

$$|f(x, y) - f(x, Y)| \leqq \max_{i=1, \cdots, n} ((n-i+1)!/(x-\xi)^{n-i+1}) |Y_i - y_i|.$$

Under these conditions and for a sufficiently small interval $\xi < x < \xi + \epsilon$ it is shown that there is at most one solution of the differential equation

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

satisfying the initial conditions $y = \eta_1$, $dy/dx = \eta_2$, $\cdots$, $d^n y/dx^n = \eta_{n+1}$ at $x = \xi$.   *F. G. Dressel* (Durham, N. C.).

Ostrowski, Alexandre.   Sur les conditions de validité d'une classe de relations entre les expressions différentielles linéaires. Comment. Math. Helv. 15, 265–286 (1943).

Under the assumption that second derivatives exist, the relations

$$(1) \quad \sum_{j=1}^{k} Q_{j\nu} A_{r\mu}^{\lambda j} = -\sum_{j=1}^{k} Q_{j\nu} A_{\mu r}^{\lambda j}, \quad \mu, \nu = 1, \cdots, n; \lambda = 1, \cdots, l,$$

between the functions $Q_{j\nu}$ and $A_{r\mu}^{\lambda j}$ imply

$$(2) \quad \sum_{j=1}^{k}\sum_{r=1}^{n} Q_{j\nu} \frac{\partial}{\partial x_r} \sum_{\lambda=1}^{l}\sum_{\mu=1}^{n} A_{r\mu}^{\lambda j} \frac{\partial z_\lambda}{\partial x_\mu} = \sum_{j=1}^{k}\sum_{r=1}^{n}\sum_{\lambda=1}^{l}\sum_{\mu=1}^{n} Q_{j\nu} \frac{\partial A_{r\mu}^{\lambda j}}{\partial x_r} \frac{\partial z_\lambda}{\partial x_\mu}.$$

The purpose of the present paper is to show (1) implies (2) under weaker conditions than the existence of second derivatives. The following conditions are shown to be sufficient: the z_λ have continuous first partial derivatives, the $A_{r\mu}^{\lambda j}$ have total differentials, and the expressions

$$\sum_{\lambda=1}^{l}\sum_{\mu=1}^{n} A_{r\mu}^{\lambda j} \frac{\partial z_\lambda}{\partial x_\mu}$$

are uniformly differentiable with respect to x_r, at the point under consideration. [A function $f(x_1, \cdots, x_n)$ is said to be uniformly differentiable with respect to x_1 at point $(a_1, \cdots, a_n)$ if

$$\frac{f(x_1, \cdots, x_n) - f(a_1, x_2, \cdots, x_n)}{x_1 - a_1}$$

tends towards a limit $f_x(a_1, \cdots, a_n)$ with $(x_1 - a_1) \to 0$, $|x_r - a_r| \leqq |x_1 - a_1|$.]   *F. G. Dressel* (Durham, N. C.).

Ostrowski, Alexandre. Sur un théorème fondamental de la théorie des équations linéaires aux dérivées partielles. Comment. Math. Helv. 15, 217–221 (1943).

If A_i, B_i, u have continuous first partial derivatives with respect to their arguments $x_1, \cdots, x_n$, E. Schmidt [Monatsh. Math. Phys. 48, 426–432 (1939); cf. these Rev. 1, 76] and others have shown that, if

$$X(u) = \sum_{i=1}^{n} A_i \partial u/\partial x_i = 0, \quad Y(u) = \sum_{i=1}^{n} B_i \partial u/\partial x_i = 0,$$

then

$$Z(u) = \sum_{i=1}^{n} [X(B_i) - Y(A_i)] \partial u/\partial x_i = 0.$$

This result follows immediately from $Z(u) = X(Y(u)) - Y(X(u))$ in case u is assumed to have continuous second partial derivatives. The present paper gives the following generalization of the results of Schmidt. If A_i, B_i, $\partial u/\partial x_i$

are continuous in the neighborhood of a point P, and A_i, B_i, $X(u)$, $Y(u)$ possess total differentials at point P, then

$$Z(u) = \sum_{i=1}^{n} [X(B_i) - Y(A_i)] \partial u/\partial x_i = X(Y(u)) - Y(X(u))$$

at the point P. The proof is more simple than those previously given.   *F. G. Dressel* (Durham, N. C.).

Sommerfeld, A.   Die ebene und sphärische Welle im polydimensionalen Raum. Math. Ann. 119, 1–20 (1943). [MF 10093]

This paper is concerned with various expansions and identities in generalized harmonics. The relations arrived from the plane and spherical wave solutions of $\Delta u + ku = 0$ and the method depends on consideration of the Green's function. The key relations are closely connected with classical results of Gegenbauer [specifically the formulae eq. 2, p. 363 and eq. 2, p. 368, cited in G. N. Watson, Theory of Bessel Functions, Cambridge, 1922].   *D. G. Bourgin.*

Brown, Herbert Kapfel.   Resolution of temperature problems by the use of finite Fourier transformations. Bull. Amer. Math. Soc. 50, 376–385 (1944). [MF 10602]

The author applies the finite Fourier transform to the solution of the linear partial differential equation

$$\frac{\partial U}{\partial t} - C_1(t)\frac{\partial^2 U}{\partial x^2} + C_2(t)U = P(x, t), \quad 0 < x < \pi; t > 0.$$

The boundary conditions considered are $U(+0, t) = C_3(t)$, $U(\pi - 0, t) = C_4(t)$, $t > 0$, while the initial condition is $U(x, +0) = F(x)$, $0 < x < \pi$. The functions $C_i(t)$, $i = 1, \cdots, 4$, $F(x)$ and $P(x, t)$ are prescribed functions. A verification of the above solution is given. The paper concludes with the solution of a similar three dimensional problem.
    A. E. Heins (Cambridge, Mass.).

de Groot, S. R.   Sur l'intégration de quelques problèmes aux limites régis par l'équation de Fourier dite "de la chaleur" au moyen de la méthode des transformations fonctionnelles simultanées. Nederl. Akad. Wetensch., Proc. 45, 643–649, 820–825 (1942). [MF 10426, 10436]

The author applies a double transformation of the type

$$(1) \quad \underset{t \ x}{LF}U(x, t) = \int_0^\infty e^{-st}dt \int_a^b K(\lambda x) U(x, t)dx$$

to the solution of a number of boundary value problems associated with (2) $u_{xx} = u_t$. Here $K(\lambda x)$ is a kernel of the "Fourier" type, that is, $\sin \lambda x$, $\cos \lambda x$ or $e^{\lambda x}$, where λ may be either discrete or continuous. The choice of the kernel and the limits a and b in (1) depend on the particular boundary value problem to be handled. Thus, if we require a solution of (2) with (3) $\lim_{x \to 0} U(x, t)$, $\lim_{x \to 0} U(x, t)$ and $\lim_{x \to 1} U(x, t)$ prescribed, one would choose $K(\lambda x) = \sin \lambda \pi x$, $\lambda = 0, 1, \cdots$, $a = 0$, $b = 1$. Then the transformation (1) will convert the equation (2) into a linear algebraic equation in

$$\underset{t \ x}{LF}U(x, t).$$

This kernel $\sin \lambda \pi x$ enjoys the property that it introduces only the prescribed functions (3) and no extraneous boundary functions. Having thus found

$$\underset{t \ x}{LF}U(x, t),$$

one may apply the proper double inverse transformation to determine $U(x, t)$. Several other boundary conditions

are considered. For example, if $\lim_{s\to 0} \partial U(x, t)/\partial x$ and $\lim_{s\to 1} \partial U(x, t)/\partial x$ are assigned, $K(\lambda x) = \cos \lambda \pi x$, $\lambda = 0, 1, 2, \cdots, a = 0$ and $b = 1$; if $\lim_{s\to 0} U(x, t)$ and $\lim_{s\to 1} \partial U(x, t)/\partial x$ are assigned, $K(\lambda x) = \sin (\lambda + \frac{1}{2})\pi x$, $\lambda = 0, 1, \cdots, a = 0, b = 1$, etc. In all of these cases $K(\lambda x)$ is so chosen as to introduce only the prescribed boundary elements.   *A. E. Heins.*

Chvedelidze, B. V. On the boundary value problem in Poincaré's theory of the logarithmic potential for a multiply connected domain. I, II. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 571–578, 865–872 (1941). (Russian. Georgian summary) [MF 10295]

The author's aim is to reduce Poincaré's boundary problem, that of finding a function $u(x, y)$, harmonic in a multiply connected domain which satisfies

$$a(s)du/dn + b(s)du/dx + c(s)u + d(s) = 0$$

along the boundary L, to a Fredholm equation and show that it has a solution. Writing u as the potential of a mass distribution along L, the problem is reduced to a singular integral equation of the form

$$a(x)\varphi(x) - \int_L (K(x, y)/(x-y))\varphi(y)dy = f(x),$$

where x and y are complex variables. Using the solution of Riemann's problem given by Gakhov [Rec. Math. [Mat. Sbornik] N.S. 2(44), 673–683 (1937)], this integral equation is then transformed into a regular Fredholm integral equation.   *František Wolf* (Berkeley, Calif.).

Nef, Walter. Über eine Verallgemeinerung des Satzes von Fatou für Potentialfunktionen. Comment. Math. Helv. 16, 215–241 (1944).

Fatou's theorem states that, if $f(z)$ is analytic and bounded in the unit circle $|z| < 1$, then $\lim_{r\to 1} f(re^{i\vartheta})$ exists for almost all ϑ in $0 \leq \vartheta \leq 2\pi$. This result has an obvious interpretation for potential functions $\Phi(x, y)$. The author shows the validity of the result for harmonic functions of arbitrarily many variables. His main theorem is as follows. If $\varphi(x_1, \cdots, x_n)$ is harmonic in the unit sphere $x_1^2 + \cdots + x_n^2 < 1$ and if the n partial derivatives $\partial\varphi/\partial x_j$ $(j = 1, \cdots, n)$ are all bounded in the unit sphere, then these derivatives converge for radial approaches from within the sphere to a point on the boundary with the possible exception of a set of points on the boundary of $(n-1)$-dimensional measure zero.

The author bases the proof upon the theory of regular hypercomplex functions

$$f(z) = \sum_1^n u_j(x_1, \cdots, x_n)e_{j_1}$$

where $z = x_1e_1 + \cdots + x_ne_n$ and $e_1, \cdots, e_n$ belong to the basis elements of a Clifford algebra with 2^n basis elements $1, e_1, \cdots, e_n, e_{12}, \cdots, e_{12\cdots n}$. The e_j satisfy $e_j^2 = 1$ $(j = 1, \cdots, n)$ and $e_j e_k = -e_k e_j$ $(j, k = 1, \cdots, n; j \neq k)$. The function f is called regular in a domain H if it possesses continuous partial derivatives of second order in H and if $\sum_1^n (\partial u/\partial x_j)e_j = 0$ holds in H. The author proves a series of results relating harmonic functions and regular functions and then proves a theorem on regular functions in the unit sphere which yields the desired result on harmonic functions. The work on regular functions is based upon the work of R. Fueter [Comment. Math. Helv. 14, 394–400 (1942); these Rev. 4, 139].   *W. T. Martin.*

Monna, A. F. Sur une classe de fonctions sous-harmoniques et des triples de fonctions harmoniques conjuguées. Nederl. Akad. Wetensch., Proc. 45, 687–689 (1942). [MF 10433]

A function $f(x, y)$, defined in a domain D, is said to be of class PL in D provided $f(x, y)$ is nonnegative and $\log f(x, y)$ is subharmonic in D. For example, the absolute value of an analytic function of a complex variable is of class PL. Many results in complex variable theory, particularly results associated with the principle of the maximum, hold also for functions of class PL. Because of the importance of functions of class PL in the differential geometry of minimal surfaces and surfaces of nonpositive Gaussian curvature, etc., a systematic development of the extension of the principle of the maximum to these functions might well be made.

The present paper is concerned with a generalization, in the direction of a theorem of F. and M. Riesz, of a result of T. Radó and the reviewer [Trans. Amer. Math. Soc. 35, 648–661 (1933)] concerning functions of class PL. It is shown that, if $f(x, y)$ is of class PL and bounded in $D: x^2 + y^2 < 1$, and if there is a set E of positive measure on $C: x^2 + y^2 = 1$, such that $\lim f(x, y) = 0$ for radial approach to points of E from inside D, then $f(x, y) = 0$. An application is made to triples of conjugate harmonic functions, that is, to sets of three functions mapping the domain of definition conformally on minimal surfaces.   *E. F. Beckenbach.*

Seward, D. M. Harmonic continuation in space. Amer. J. Math. 66, 255–267 (1944). [MF 10571]

The author gives an extension to three-dimensional space of Hadamard's theorem on harmonic continuation in the plane [Mémoires présentés par divers savants à l'Académie des Sciences de l'Institute de France (2) 33, 23–27 (1908)]. A set σ of points $P(x, y, z)$ in space is said to be an analytic surface set if to each point p of σ there corresponds a function $E_p(P) = E_p(x, y, z)$ which, in some sphere about p, is analytic, has a nonvanishing gradient ∇E_p and vanishes on, and only on, σ. A function $U(P)$ defined on a set σ of points in space is said to be analytic on σ if to each point p of σ there corresponds a function $V(P)$ which in some sphere about p is analytic and coincides with $U(P)$ on σ. The paper is devoted to a discussion of the following theorem. Let D be a domain in space with boundary d; let the frontier of D contain an analytic surface set σ no point of which is at zero distance from $d - \sigma$; let $U(P) = U(x, y, z)$ be harmonic in D and such that either (a) $U(P)$ is continuous on $D + \sigma$ and analytic on σ or (b) U, U_x, U_y, U_z coincide in D with functions continuous on $D + \sigma$, and $\partial U/\partial n$ (the outer normal derivative of U on σ) is analytic on σ. Then there exists a function $U^*(P)$, harmonic in a domain D^* containing $D + \sigma$ and coincident with $U(P)$ in D.

F. W. Perkins (Hanover, N. H.).

Vekua, Ilja. On the solution of the equation $\Delta u + \lambda^2 u = 0$. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 307–314 (1942). (Georgian. Russian summary) [MF 10315]

A function $u(x_1, x_2, \cdots, x_n)$ having continuous second derivatives in its domain of definition D is said to be metaharmonic in D provided it satisfies the differential equation $\Delta u + \lambda^2 u = 0$ in D, where Δ is the Laplace operator and λ is constant. It is shown that metaharmonic functions can be characterized in terms of harmonic functions by means of an integral equation of Volterra type. Accordingly metaharmonic functions can be given characteristic series ex-

pansions in terms of polar coordinates $(r, \varphi_1, \cdots, \varphi_{n-1})$. Results such as the following are consequences of the series expansion. If u is metaharmonic throughout Euclidean n-space and if

$$\lim_{r \to \infty} [r^{(n-1)/2}u] = 0,$$

then $u = 0$. From the integral equation there is obtained, for an arbitrary function metaharmonic in a star-shaped region with smooth boundary, an expression in terms of a potential of a double layer. Dirichlet and Neumann problems are discussed.   E. F. Beckenbach (Austin, Tex.).

Bremekamp, H. Sur l'unicité des solutions de certaines équations aux dérivées partielles du quatrième ordre. Nederl. Akad. Wetensch., Proc. **45**, 546–552 (1942). [MF 10419]

Bremekamp, H. Sur l'existence et la construction des solutions de certaines équations aux dérivées partielles du quatrième ordre. Nederl. Akad. Wetensch., Proc. **45**, 675–680 (1942). [MF 10431]

The first paper is concerned with the uniqueness of the solutions of the equation

$$\alpha \Delta\Delta u + 2\beta \Delta u + \gamma u = 0, \quad \Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2; \; \alpha \neq 0,$$

in a region D when u and Δu, or u and the normal derivative $\partial u/\partial n$, are given at each point of the closed boundary C of D. The solutions are shown to be unique for any bounded region D if $\alpha\gamma - \beta^2 > 0$ and also unique for $\alpha\gamma - \beta^2 \leq 0$ if the region D is sufficiently small (α, β, γ are holomorphic functions of x and y). Using a method of approximation similar to that employed by Picard for the equation

$$\Delta u + \alpha \partial u/\partial x + \beta \partial u/\partial y + \gamma u + f = 0,$$

the second paper shows how to construct the unique solutions mentioned in the first paper.   F. G. Dressel.

Special Functional Equations

Ghermanescu, Michel. Sur une équation fonctionnelle qui caractérise des polynomes. Mathematica, Timişoara **19**, 148–158 (1943). [MF 9962]

The paper presents extensions of results of Anghelutza [Bull. Sci. Math. (2) **61**, 357–360 (1937)] and Popoviciu [Mathematica, Cluj **10**, 194–208 (1935)], where polynomials are characterized by vanishing of their differences of sufficiently high order. The author considers the functional equation

$$(1) \qquad \Delta_{b_1 \cdots b_n}^{a_1 \cdots a_n} f(x, y) = 0,$$

where the operator on the left hand side is a "mixed compound" difference, defined by

$$\Delta_{b}^{a} f(x, y) = f(x+a, y+b) - f(x, y), \quad \Delta_{b_1 \cdots b_n}^{a_1 \cdots a_n} = \Delta_{b_n}^{a_n}(\Delta_{b_1 \cdots b_{n-1}}^{a_1 \cdots a_{n-1}}).$$

The author studies relations between the a_i, b_i and polynomials of degree not less than n satisfying (1). The first theorem given may serve as an illustration. The most general solution of (1), which is linearly measurable (in x and y separately), and in which the a_i, b_i satisfy the relation

$$A_0 a_1 a_2 \cdots a_n + A_1 \sum a_1 a_2 \cdots a_{i-1} b_{i+1} \cdots a_n$$
$$+ A_2 \sum\sum a_1 \cdots a_{i-1} b_{i+1} \cdots a_{j-1} b_{j+1} \cdots a_n$$
$$+ \cdots + A_n b_1 \cdots b_n = 0,$$

is an nth degree polynomial of the form

$$P(x, y) = A_0 x^n + \binom{n}{1}A_1 x^{n-1}y + \cdots + \binom{n}{n}A_n y^n$$
$$+ (\text{arbitrary polynomial of degree } n-1).$$

The reviewer has been unable to follow the proofs or to attribute a precise meaning to the theorems given, as the author consistently neglects to indicate how far his variables are arbitrary or subject to conditions, which are not stated, or to list the arguments of his functions completely.

F. John (Aberdeen, Md.).

Klimpt, Werner. Note über eine lineare homogene Differenzen-Differentialgleichung. Arch. Math. Wirtsch.-Sozialforschg. **6**, 34–42 (1940). [MF 10651]

This paper considers the equation

$$(1) \qquad dz/dt = az(t) - cz(t-\tau),$$

in which a, c and τ are constants. Letting $K = e^{a\tau}/c\tau$, and v be a solution of (2) $e^v = Kv$, it is seen that $z = e^{rt}$ is a solution of (1) if $r = a - v/\tau$. The real and complex solutions of (2) are considered for real values of K, and graphs are given for the determination of the roots of (2). In addition to infinitely many complex roots, there are two real distinct roots when $K > e$, no real roots when $0 < K < e$ and one negative real root when $K < 0$. In the special case of the coincident real roots, the particular solutions of (1) are $z = e^{r_1 t}$ and $z = te^{r_1 t}$, where $r_1 = a - 1/\tau$.   D. Moskovits.

Nath Sarma, Prithvi. On the differential equation $f'(x) = f(1/x)$. Math. Student **10**, 173–174 (1942). [MF 9982]

The equation (1) mentioned in the title is solved by means of linear combinations of functions $x^\rho + \lambda x^{\rho'}$, where ρ and $\rho' = r - \rho$ run through the roots of the equation

$$\varphi(y) = y(y-1)^2(y-2)^2 \cdots (y-r+1)^2(y-r) - (-1)^r = 0$$

and $\lambda = \rho(\rho-1) \cdots (\rho-r+1)$ (each root shall occur in exactly one pair ρ, ρ'). The author's contention that his solution is complete seems to follow from the fact that each solution of (1) must satisfy a certain linear differential equation of order $3r-1$ whose solutions are linear combinations of functions x^ζ, where ζ runs through the roots of

$$(y-r-1)(y-r-2) \cdots (y-2r+1)\varphi(y) = 0.$$

An explicit solution is given for $r = 2$. The case $r = 1$ was previously discussed by L. Silberstein [Philos. Mag. (7) **30**, 185–186 (1940); these Rev. **2**, 134].   P. Scherk.

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van der Corput, J. G. A remarkable family. Nederl. Akad. Wetensch., Proc. **45**, 129–135, 217–224, 327–334 (1942). [MF 10379, 10389, 10398]

This paper is concerned with the solution of functional equations of the form

$$g_\rho(x, y) = 0, \quad y_{\nu\kappa} = f_\nu(l_\kappa(x)) \quad (\rho, \nu = 1, \cdots, n; \kappa = 1, \cdots, k \geq 2),$$

where the g_ρ are n given functions of the $t+kn$ variables $(x_\tau, y_{\nu\kappa})$, each of the k symbols $l_\kappa(x)$ stands for a set of t given functions of the t variables (x_τ), and the $f_\nu(x)$ are n functions of the t variables (x_τ) which are to be determined by the given equations. The first part gives sufficient conditions for the existence and uniqueness of an analytic solution in the case $t = 1$, using the method of dominant functions. The last two parts treat the case of general t, and the situation is then much more complicated.

L. M. Graves (Chicago, Ill.).

Popovici, C. Topologie fonctionnelle. Mathematica, Timişoara **19**, 119–125 (1943). [MF 9959]

If $q(z)$ is not an odd function of the complex variable z, the equation $f(z) - f(-z) = q(z)$ cannot be solved for a single-valued $f(z)$. On the other hand, multiple valued solutions are given by the author where the proper determination of $f(-z)$ must be matched with a given determination of $f(z)$. [Minor corrections are necessary in his solution.] More generally, equations involving derivatives or integrals are considered and the following general method of solution proposed. If the complex plane is divided into two halves symmetric in the origin, say red and white, $f(z)$ may be defined arbitrarily in the red half and defined in the white half so as to satisfy the equation if z is in the red and $-z$ in the white. Then f is defined in a second copy of the red so as to satisfy the equation if z is in the white and $-z$ is in the second copy of the red; and so on. The case where $-z$ is replaced by a more general function of z is also taken up. The method would not seem easily to yield solutions in case $f(z)$ is required to be continuous or differentiable.

A. B. Brown (Flushing, N. Y.).

Integral Equations

Zaanen, A. C. Ueber die Existenz der Eigenfunktionen eines symmetrisierbaren Kernes. Nederl. Akad. Wetensch., Proc. **45**, 973–977 (1942). [MF 10447]

A proof of the existence of a characteristic number for a symmetrizable kernel was given by Marty in 1910. The present note is another proof following closely the plan used by Kellogg in the corresponding result for symmetric kernels [Math. Ann. **86**, 14–17 (1922)]. However, the result is restricted to such kernels for which the closure of the symmetrizing kernel implies the closure of the original symmetrizable kernel. Such is the case, for example, for kernels of polar type.   *I. A. Barnett* (Cincinnati, Ohio).

Scherman, D. Sur la réduction d'une classe de problèmes à l'équation intégrale de Fredholm. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 603–605 (1941). [MF 9610]

The author considers the problem

$$\sum (a_{ek}\varphi_k + b_{ek}\varphi_e) = f_k(t)$$

$(k = 1, \cdots, n; e = 1, \cdots, 2n; t$ on $L; a, b, f$ suitably differentiable), where the $\varphi_e(z)$ $(z = x + iy)$ are to be determined as functions regular in a simply connected bounded domain S, of which L is the boundary (satisfying suitable conditions of "smoothness"). The substitution

$$\varphi_k(z) = \frac{1}{2\pi i}\int_L \frac{\omega_k(t)}{t - z}\,dt$$

leads simply to a problem in Fredholm integral equations (F), presenting advantages over the usual potential theoretic methods. Solubility of (F) is to be considered in each case as it is confronted.   *W. J. Trjitzinsky.*

Cameron, R. H. and Martin, W. T. An expression for the solution of a class of non-linear integral equations. Amer. J. Math. **66**, 281–298 (1944). [MF 10573]

Let $G(t, \xi, u)$ be continuous in $0 \le t, \xi \le 1, -\infty < u < \infty$, and satisfy $|G(t, \xi, u_1) - G(t, \xi, u_2)| \le M|u_1 - u_2|$. Put

$$T[x(.)|t] = x(t) - \int_0^1 G(t, \xi, x(\xi))d\xi,$$

where $x(t) \epsilon C = C_0[0, 1]$, the class of functions continuous on $[0, 1]$ and vanishing for $t = 0$. If $y(t)\epsilon C$, it is well known that the equation $T[x(.)|t] = y(t)$ has a unique solution in the same class. The authors show how this solution may be obtained by carrying out an averaging process over the space C with the aid of the Wiener integral [cf. Acta Math. **55**, 117–258 (1930), especially pp. 214–224]. They arrive at the expression

$$x_0(\tau) = \text{l.i.m.}_{\rho \to \infty} \frac{\int_C \exp\left[-\rho\int_0^1 \{T[x(.)|t] - y(t)\}^2 dt\right]x(\tau)d_w x}{\int_C \exp\left[-\rho\int_0^1 \{T[x(.)|t] - y(t)\}^2 dt\right]d_w x},$$

where the l.i.m. is taken in the $L_2(0, 1)$ sense and the two integrals are averages in the Wiener sense taken over C. The authors give a brief account of the Wiener integral which they generalize by replacing the weighting factor $e^{-\lambda\rho}$ by a more general function $E(\lambda, \rho)$ having certain specified properties. They also replace $T[x(.)|t]$ by a general operator $F[x(.)|t]$ which takes C into itself in a one-to-one manner and which together with its inverse is continuous in the L_2-sense, the L_2-bound of the inverse satisfying

$$\left(\int_0^1 \{F^{-1}[x(.)|t]\}^2 dt\right)^{\frac12} \le KE\left(\int_0^1 x^2(t)dt, -A\right)$$

for a suitable fixed choice of A and K. The authors prove that, if $F[x(.)|t]$ is Wiener-Lebesgue measurable in $x(.)$ and t, then the solution of $F[x(.)|t] = y(t)$ is given by $F^{-1}[y(.)|\tau]$

$$= \text{l.i.m.}_{\rho \to \infty} \frac{\int_C E\left(\int_0^1 \{F[x(.)|t] - y(t)\}^2 dt, \rho\right)x(\tau)d_w x}{\int_C E\left(\int_0^1 \{F[x(.)|t] - y(t)\}^2 dt, \rho\right)d_w x}.$$

The special result listed above then follows by observing that $e^{-\lambda\rho}$ satisfies the conditions imposed on $E(\lambda, \rho)$ and that $T[x(.)|t]$ is an admissible operator $F[x(.)|t]$.

E. Hille (New Haven, Conn.).

Godefroy, Marcel. Sur la résolution au moyen de fonctions holomorphes de certaines équations intégro-différentielles. C. R. Acad. Sci. Paris **213**, 336–337 (1941). [MF 9162]

A metric M_l $(l \le 1)$ is defined by taking as the norm of a function $f(\lambda) = \sum a_n\lambda^n$, holomorphic for $|\lambda| < 1$, the expression $|f(\lambda)| = \sum |a_n|l^n$. The distance of two functions is the norm of their difference. The author announces these results. (1) If $\sum |u_n(\lambda)|$ is convergent the series $\sum u_n(\lambda)$ converges on a circle of center 0 and radius l and has in M_l a norm equal at most to $\sum |u_n|$. (2) The set of functions $f(\lambda)$ holomorphic for $|\lambda| < 1$ with either $|a_n| \le G$ or $\sum |a_n| \le G$ is complete in itself in each metric M_l. (3) If $f(\lambda)$ has a norm $d \le 1/l$ in M_l and if $|a_n|R^n \le G$ for $l < R \le 1$, then $f'(\lambda)$ has for a norm in M_l a number less than $kd|\log d|$, where k depends only upon G, R, l. (4) Let $f(\lambda, \mu)$ be holomorphic for $|\lambda|$ and $|\mu| < 1$ and d the norm of $f(\lambda, \lambda)$ in M_l with $|\mu_1|$ and $|\mu_2| < l < 1$. Then $F(\lambda)$ is holomorphic for $|\lambda| < 1$ and has in M_l a norm less than kd, k depending upon $l, |\mu_1|$ and $|\mu_2|$, where

$$F(\lambda) = \int_\Gamma \frac{f(\lambda, \mu)(-(\mu - \mu_1)(\mu - \mu_2))^{\frac12}}{\lambda - \mu}d\mu.$$

Here Γ is a circle of center 0, radius R, and $|\mu_1|, |\mu_2|$ and $|\lambda| < R < 1$.

The author applies these theorems to obtain the following result. Let $H(\lambda)$ and $\zeta(\lambda)$ be holomorphic for $|\lambda| < 1$, λ_1 and λ_2 real and less than one in modulus. Suppose that $\zeta(\lambda)$ is bounded and univalent. Then there cannot be more than one function $\zeta(\lambda, t)$ satisfying

$$\left[\frac{\partial}{\partial t}\zeta(\lambda, t)\right] = \frac{1}{4\pi i}\int_\Gamma \frac{H(\lambda')(-(\lambda'-\lambda_1)(\lambda'-\lambda_2))^{\frac{1}{2}}}{\zeta(\lambda, t) - \zeta(\lambda', t)}d\lambda'$$

and which is holomorphic in λ as well as its derivative with respect to t for $|\lambda| < 1$, $t_0 < t < t_1$ and which is bounded and univalent for t fixed.   *M. S. Robertson.*

Wick, G. C. Über ebene Diffusions-probleme. Z. Phys. **121**, 702–718 (1943). [MF 10057]

In this paper a new method is developed for solving the integro-differential equation

$$(1) \qquad (u\partial/\partial x + 1)f(x, u) - (\sigma/2)\int_{-1}^{+1} f(x, v)dv = 0,$$

σ a constant. The principle of the method consists in expressing the integral occurring in this equation as a sum according to Gauss' formula for numerical integration in the form

$$(2) \qquad \int_{-1}^{+1} f(x, v)dv \simeq \sum_i p_i f(x, u_i),$$

where the u_i's are the zeros of a Legendre polynomial of even order and the p_i's are certain weight factors. On this approximation, equation (1) splits up into a system of ordinary linear equations for the unknown functions $f(x, u_i)$. These equations can be explicitly solved in the form

$$(3) \qquad f(x, u_i) = \sum_a \frac{L_a}{1 - k_a u_i}e^{-k_a x},$$

where the L_a's are certain constants of integration and the k_a's are the roots of the equation

$$(4) \qquad 1 = \frac{\sigma}{2}\sum_i \frac{p_i}{1 - ku_i}.$$

The constants L_a are then to be determined by using the boundary conditions appropriate to the problem on hand. In the paper certain convenient methods for the practical determination of these constants L_a are outlined. Also, the extension of the method to solving nonhomogeneous equations in which on the right hand side of equation (1) we have a known function of fairly simple form is also considered.

It is evident that, in the foregoing method of solution, successively higher approximations can be obtained by choosing the u_i's to correspond to the zeros of higher and higher order Legendre polynomials (of even degree). However, it appears that the method is a rapidly converging one, and in one case for which the exact solution was known the third approximation (derived from the zeros of $P_6(u)$) already agrees very closely with the known solution.
                                    S. Chandrasekhar (Williams Bay, Wis.).

NUMERICAL AND GRAPHICAL METHODS

✻**Tables of Lagrangian Interpolation Coefficients.** Prepared by the Federal Works Agency, Work Projects Administration; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. Columbia University Press, New York, 1944. xxxvi+392 pp. $5.00.

The classical methods of practical interpolation were developed at a time when no modern aids for computation were available. Accordingly they are no longer practical. The operation of a scalar product is particularly simple to perform on any modern computing machine and that is why the Lagrange interpolation formula is particularly well adapted for modern computation. The lack of extensive tables of the Lagrange interpolation coefficients has been badly felt and the present volume fills this gap. Among the many useful tables prepared by the same Project, the present one will find an especially grateful public.

The interpolation coefficients are given to 10 decimal places for the following intervals:

Three-point $[-1(.0001)1]$.
Four-point $[-1(.001)0(.0001)1(.001)2]$.
Five-point $[-2(.001)2]$.
Six-point $[-2(.01)0(.001)1(.01)3]$.
Seven-point $[-3(.01)-1(.001)1(.01)3]$.
Eight-point $[-3(.1)0(.001)1(.1)4]$.
Nine-point $[-4(.1)4]$. Ten-point $[-4(.1)5]$.
Eleven-point $[-5(.1)5]$.

Moreover, for convenience, three-, $\cdots$, eight-point values are repeated for p at intervals of .1, and these coefficients are also given in fractional form for p in multiples of 1/12.

Finally the corresponding integration coefficients are also tabulated.   *W. Feller* (Providence, R. I.).

Salzer, Herbert E. Table of coefficients for inverse interpolation with central differences. J. Math. Phys. Mass. Inst. Tech. **22**, 210–224 (1943). [MF 9780]

To facilitate the laborious task of inverse interpolation H. T. Davis [Tables of the Higher Mathematical Functions, vol. I, The Principia Press, Bloomington, Indiana, 1937, pp. 82–83] inverted Everett's interpolation formula and obtained the inverse formula

$$(1) \quad p = m + \binom{m}{3}\frac{\delta_0^2}{\Delta} + \binom{m+1}{3}\frac{\delta_1^2}{\Delta} + A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta}$$
$$+ C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta}$$

plus terms of sixth and higher orders. Here

$$p = (x - x_0)/(x_1 - x_0), \quad m = (y - y_0)/(y_1 - y_0), \quad \Delta = y_1 - y_0,$$

while $\delta_0^2, \delta_0^4, \cdots, \delta_1^2, \delta_1^4, \cdots$, denote the central differences on the same lines as y_0 and y_1, respectively. The coefficients $A(m)$, $B(m)$, $\cdots$ are polynomials of fifth degree in the parameter m.

The author's principle table gives the values to ten decimal places for these five coefficients $A(m), \cdots, E(m)$ at intervals of 0.001 for the range $m = 0$ to $m = 1$. A short additional table gives the values at intervals of 0.1 for the ten coefficients of the terms of sixth order. Values for the second order coefficients are not given as these are found in tables for direct interpolation by Everett's formula.
                                    W. E. Milne (Corvallis, Ore.).

Heatley, A. H. **A short table of the Toronto function.** Trans. Roy. Soc. Canada. Sect. III. **37,** 13–29 (1943). [MF 9974]

This paper deals with the numerical computation of a modified form of the $T(m, n)$ function discussed in an earlier paper [University of Toronto Studies, Mathematical Series, no. 7, Toronto, 1939; these Rev. **2,** 45]. The author now considers the "Toronto" function defined as

$$(1) \quad T(m, n, r) = r^{2n-m+1} e^{-r^2} \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)} {}_1F_1(\frac{1}{2}m+\frac{1}{2}; n+1; r^2).$$

Various properties of the function are described, and its values in terms of known functions for particular values of m and n are listed in table I. The remaining tables give numerical values of the hypergeometric function ${}_1F_1(\alpha; \gamma; x)$, in ranges not previously tabulated, and of (1) for positive values of r and the values $m=0, \pm\frac{1}{2}, 1$; $n=0, \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2$. *M. C. Gray* (New York, N. Y.).

Bose, Purnendu Kumar. **On confluent hypergeometric series.** Sankhyā **6,** 407–412 (1944). [MF 10621]

Numerical values of the confluent hypergeometric series ${}_1F_1(\alpha, \rho, x)$ are tabulated for integral values of the parameters α and ρ ($1\leq\rho\leq4$; $2\leq\alpha\leq25$), and for selected values of x ($2\leq x\leq13.5$). *M. C. Gray* (New York, N. Y.).

Mathematical Tables Project. **Table of**
$$f_n(x) = (n!/(x/2)^n) J_n(x).$$

J. Math. Phys. Mass. Inst. Tech. **23,** 45–60 (1944). [MF 10117]

The function $J_n(x)$ for even moderately large values of n is very small over a considerable range of x near $x=0$, so that tables of $J_n(x)$ to a fixed number of decimal places give few significant figures. To surmount this difficulty the present table gives the function

$$f_n(x) = (n!/(x/2)^n) J_n(x).$$

Values of $f_n(x)$ are given to 9 decimal places for each integral n from $n=2$ to $n=20$ inclusive. The range for x is 0 to 10 with intervals of 0.05 for smaller values of x and n, and 0.1 for larger values. Modified second central differences are tabulated alongside the entries. *W. E. Milne.*

Smith, D. B., Rodgers, L. M. and Traub, E. H. **Zeros of Bessel functions.** J. Franklin Inst. **237,** 301–303 (1944). [MF 10235]

The two tables [pp. 302 and 303] give eight positive values of x for which $J_0(x)$, $J_0'(x)$ and $J_1'(x)$ are zero, seven positive values for which $J_1(x)$, $J_2(x)$, $J_2'(x)$ and $J_3'(x)$ are zero, six positive values for which $J_3(x)$, $J_4(x)$, $J_4'(x)$ and $J_5'(x)$ are zero, five positive values for which $J_5(x)$, $J_6(x)$, $J_7(x)$, $J_6'(x)$, $J_7'(x)$ and $J_8'(x)$ are zero, four positive values for which $J_8(x)$, $J_9(x)$, $J_9'(x)$ and $J_{10}'(x)$ are zero, three positive values for which $J_{10}(x)$, $J_{11}(x)$, $J_{12}(x)$, $J_{11}'(x)$, $J_{12}'(x)$ and $J_{13}'(x)$ are zero, two positive values for which $J_{13}(x)$, $J_{14}(x)$, $J_{15}(x)$, $J_{14}'(x)$, $J_{15}'(x)$, $J_{16}'(x)$ and $J_{17}'(x)$ are zero, one positive value for which $J_{16}(x)$, $J_{17}(x)$, $J_{18}(x)$, $J_{19}(x)$, $J_{18}'(x)$, $J_{19}'(x)$, $J_{20}'(x)$, $J_{21}'(x)$ and $J_{22}'(x)$ are zero. The number of decimal places is generally four but is five in forty-two cases. The tables were prepared for use by the Research Division of the Philco Engineering Department. *H. Bateman* (Pasadena, Calif.).

Vašakidze, D. R. **Approximate formulae for elliptic integrals of the second kind.** Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2,** 597–600 (1941). (Russian. Georgian summary) [MF 10298]

Saibel, Edward. **A rapid method of inversion of certain types of matrices.** J. Franklin Inst. **237,** 197–201 (1944). [MF 10118]

The method applies in particular to matrices with zero elements everywhere except in the main diagonal and in the diagonals just above and below the main diagonal. The matrix is partitioned and the inversion reduced to the inversion of certain matrices of lower order. [Cf. H. Hotelling, Ann. Math. Statistics **14,** 1–34 (1943); these Rev. **4,** 202.] *W. Feller* (Providence, R. I.).

Hotelling, Harold. **Further points on matrix calculation and simultaneous equations.** Ann. Math. Statistics **14,** 440–441 (1943). [MF 9848]

Minor additions and errata to the author's paper in the same Ann. **14,** 1–34 (1943); cf. these Rev. **4,** 202. *W. Feller* (Providence, R. I.).

Dwyer, Paul S. **A matrix presentation of least squares and correlation theory with matrix justification of improved methods of solution.** Ann. Math. Statistics **15,** 82–89 (1944). [MF 10241]

This paper is a compact matrix formulation of the determination of the constants by least squares in linear regression systems. It includes the author's improved methods of solution of the normal equations and of finding values of associated numerical quantities. It is remarked that the essential feature of the Doolittle solution of a system of linear equations is the factorization of the matrix of coefficients and that from any such factorization the essential results follow. This factorization can be made to give two equal triangular matrix factors and this leads to a method in some respects superior to the Doolittle solution. This will be developed in detail in a later paper. *C. C. Craig.*

Vernotte, Pierre. **Détermination, par la condition de moindre imprécision d'une formule dépendant linéairement de paramètres, destinée à la représentation d'une courbe expérimentale.** C. R. Acad. Sci. Paris **216,** 33–35 (1943). [MF 10000]

Vernotte, Pierre. **Représentation d'une courbe expérimentale, dans le cas général, par la condition de moindre imprécision.** C. R. Acad. Sci. Paris **216,** 148–150 (1943). [MF 10008]

Vernotte, Pierre. **Sur les systèmes d'équations auxquels conduit la méthode de la moindre imprécision.** C. R. Acad. Sci. Paris **215,** 289–291 (1943). [MF 10050]

Vernotte, Pierre. **Détermination, par la condition de moindre imprécision, des coefficients d'une formule représentant une courbe expérimentale, où ils figurent linéairement.** C. R. Acad. Sci. Paris **215,** 568–570 (1942). [MF 10189]

Let y_i ($i=1, 2, \cdots, n$) be experimentally observed ordinates of a curve $y=f(x, A, B, C)$, where A, B, C, etc. are coefficients to be determined so as to give the "best" fit in some sense. Also let $Q=\varphi(A, B, C)$ be some quantity which it is the object of the experiment to determine. By the "condition of least inaccuracy" (moindre imprécision) the author means that A, B, C, etc. shall be so determined that to an error δy_i in y_i shall correspond the least maximum error $|\delta_i Q|$ for x in the given range of values.

Subject to very considerable limitations he succeeds in expressing the coefficients A, B, C, etc. in terms of the y_i so as to satisfy the condition of least inaccuracy in several simple cases. The limitations are: (a) the abscissas of the given points are equally spaced; (b) A, B, C, etc. are as-

sumed linear in the y_i; (c) $f(x, A, B, C)$ is a polynomial in x of low degree (usually linear) or a function of the form $f(Ax+B)$; (d) $f(x, A, B, C)$ is linear in A, B, C, etc.; (e) Q is linear in A, B, C, etc. Conditions (d) and (e) can be lightened by resort to successive approximations.

Typical of the results obtained is this one for n odd $(n=2N+1)$, $y=Ax+B$, where A and/or B are the quantities desired. Taking the abscissas at $x=0$, ± 1, $\cdots$, $\pm N$ he gets

$$A=(1/N(N+1))[-y_1-\cdots-y_N+y_{N+2}+\cdots+y_{2n+1}],$$
$$B=(1/(2N+1))[y_1+y_2+\cdots+y_{2N+1}].$$

On the other hand, if y is the quantity desired, he finds that

$$y=\varphi_1 y_1+\varphi_2 y_2+\cdots+\varphi_n y_n,$$

where the φ's are linear expressions in x whose extrema in $(x_1 x_n)$ are equal in absolute value. Explicit formulas are given for $n=2, 3, \cdots, 12$. 　　　　　*W. E. Milne.*

Vickery, C. W. Cyclically invariant graduation. Econometrica **12**, 19–25 (1944). [MF 10071]

For a smoothing formula based on a symmetric weight function the author derives the factor by which the smoothing process magnifies the amplitude of any sinusoidal term, in agreement with H. Labrouste [C. R. Acad. Sci. Paris **184**, 259–261 (1927)]. He considers that the use of such smoothing formulas is justified by supposing the sequence to have arisen from a parent sequence by a process of disturbing the abscissas by independent random amounts. In nature such disturbances are usually not independent but tend to vary smoothly with the abscissa; in fact, if they were independent, they could lead to a multiple-valued function. 　　　　　*A. Blake* (Washington, D. C.).

Boyer, John F. Osculatory interpolation in practice. Record. Amer. Inst. Actuar. **32**, 83–96 (1943). [MF 10590]

Discussion of the paper in the same Record **31**, 337–350 (1942); these Rev. **4**, 283.

Chadaja, F. G. On the problem of numerical integration of ordinary differential equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 601–608 (1941). (Russian. Georgian summary) [MF 10299]

This paper gives a set of formulas for solving numerically the differential equation

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$$

with initial values $y^{(k)}=y_0^{(k)}$, $k=0, 1, \cdots, n-1$. The point of departure is Taylor's series with remainder in the form

$$y^{(k)}(x)=y^{(k)}(a)+y^{(k+1)}(a)(x-a)+\cdots$$
$$+\frac{y^{(n-1)}(a)(x-a)^{n-k-1}}{(n-k-1)!}+\frac{1}{(n-k-1)!}\int_a^x (x-z)^{n-k-1}y^{(n)}(z)dz.$$

In the integral on the right, the author replaces $y^{(n)}(z)$ by Newton's backward interpolation formula, thus obtaining a formula which we call A. Similarly, using Newton's forward interpolation formula he gets a formula which we call B and by a combination of these two he gets a third, C. One or more of these formulas A, B, C can be used to calculate by successive approximations a few entries in a table of values near the initially given point x_0. Then, by a simple modification of the foregoing procedure, he develops a pair of formulas D, E that can be employed for the step-by-step continuation of the table once a few starting values have been obtained. The question of convergence of the

successive approximations is examined and a numerical example presented. Unfortunately the detailed steps of the computation are omitted. As the formulas look somewhat clumsy, it would be very helpful to show just how the work is to be arranged for rapid and systematic calculation.
　　　　　W. E. Milne (Corvallis, Ore.).

Mikeladze, Š. E. On the approximate integration of linear differential equations with discontinuous coefficients. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **3**, 633–639 (1942). (Russian. Georgian summary) [MF 10318]

The author's procedure is simply to replace the linear differential equation by a linear difference equation in the usual manner throughout each interval of continuity, while at points of discontinuity he replaces right hand derivatives by suitable expressions in terms of forward differences and left hand derivatives by backward differences. He applies his method to find the critical force in the problem of stability of a rod under longitudinal compression, where the rod consists of three segments with different cross sections. He obtains the critical force by finding the least root of the characteristic determinant of the set of homogeneous linear equations and checks his result with the rigorous solution.
　　　　　W. E. Milne (Corvallis, Ore.).

Kuehni, H. P. and Peterson, H. A. A new differential analyzer. Elec. Engrg. **63**, 221–228 (1944).

This paper describes a new differential analyzer, built by the General Electric Company at Schenectady, with 14 integrators, 4 input-output tables and two output tables. The machine can be divided into two parts to solve two equations simultaneously. The construction follows that of the original M.I.T. and University of Pennsylvania analyzers generally, the chief novelty being the use of a servo system controlled by polarized light in place of torque amplifiers. This eliminates output load on the integrator wheels, and integrator slip is reduced to the order of 0.02%.
　　　　　C. E. Shannon (New York, N. Y.).

Greville, T. N. E. Short methods of constructing abridged life tables. Record. Amer. Inst. Actuar. **32**, 29–43 and Discussion 408–418 (1943). [MF 10588]

Greville, T. N. E. "Census" methods of construction of mortality tables and their relation to "insurance" methods. Record. Amer. Inst. Actuar. **32**, 125–130 (1943). [MF 10591]

Discussion of the paper in the same Record **31**, 367–373 (1942); these Rev. **4**, 281.

Lang, Kermit. Analysis of net premium formulas for the income endowment policy. Record. Amer. Inst. Actuar. **32**, 156–170 (1943). [MF 10592]

Discussion of the paper in the same Record **31**, 398–405 (1942); these Rev. **4**, 281.

Shannon, S. A theory of automatic premium-loan approximations: Formulas derived and compared. Record. Amer. Inst. Actuar. **32**, 74–82 and Discussion 427–435 (1943). [MF 10589]

Hain, Kurt. Die Verwendung des Gelenkvierecks als Rechengetriebe. Z. Instrumentenkunde **63**, 170–180 (1943). [MF 10082]

The author discusses the problem of constructing a four-bar-linkage for mechanizing a prescribed motion.
　　　　　W. Feller (Providence, R. I.).

MECHANICS

Hydrodynamics, Aerodynamics, Acoustics

Dolidze, D. E. Über das nichtlineare Problem der Hydrodynamik im Raume von drei Dimensionen. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 499–506 (1941). (Russian. Georgian and German summaries) [MF 10308]

Oudart, Adalbert. Problème des sillages. Validité des solutions. C. R. Acad. Sci. Paris **213**, 679–682 (1941). [MF 9651]

Based on the results of H. Villat the author continues his work [cf. also C. R. Acad. Sci. Paris **214**, 149–151 (1942); these Rev. **4**, 175] on the investigation of the free boundary lines of the wake due to a sharp-nosed body. Results dealing with the curvature, the number of points of inflection and conditions under which the free lines are self-intersecting are presented. *A. Gelbart* (Syracuse, N. Y.).

Bergman, Stefan. A formula for the stream function of certain flows. Proc. Nat. Acad. Sci. U. S. A. **29**, 276–281 (1943). [MF 9021]

Using the hodograph method, the author defines two operators which generate solutions of the equations of motion of compressible fluids. The first operator transforms a Taylor series of a complex variable into a univalent complex function $\varphi + i\psi$, where φ and ψ are the potential and the stream function of a plane flow of a compressible fluid. [The same solutions were obtained independently by Bers and Gelbart, Quart. Appl. Math. **1**, 168–188 (1943); these Rev. **5**, 25.] The second operator is of the form

$$\psi(v, \theta) = \Im \left\{ H(v) \left[g(Z) + \sum Q^{(n)}(v) \int_0^Z \cdots \int_0^Z g(Z_n) dZ_n \cdots dZ_1 \right] \right\},$$

$Z = \lambda(v) + i\theta$. Here $ve^{i\theta}$ denotes the velocity vector, $H(v)$, $Q^{(n)}(v)$, $n = 1, 2, \cdots,$ $\lambda(v)$ are fixed functions, g is an arbitrary analytic function. This operator, acting directly on an arbitrary g, yields more general types of solutions than the former. Using orthogonal functions, the author gives a formula for the stream functions of flows which, in the case of a simplified pressure density relation $p = A + \sigma/\rho$ with constant A and σ, are bounded by segments of straight lines and by free boundaries. *A. Weinstein.*

Strang, J. A. Self-superposable motion of viscous incompressible fluid referred to rotating axes. Proc. Benares Math. Soc. (N.S.) **4**, 9–18 (1943). [MF 10343]

A fluid velocity is self-superposable if $\mathbf{U} \times (\nabla \times \mathbf{U}) = \nabla \chi$, where $\mathbf{U}$ is the velocity vector and χ is any function of x, y, z, t. Let the hydrodynamical equations be referred to axes rotating about OZ with constant angular velocity $\frac{1}{2}\Omega$, where Ω^2 is assumed negligible. The author obtains self-superposable solutions of these equations having characteristics of tides and dust whirls. He then generalizes these results to obtain approximate, non-self-superposable solutions representing tornados and waterspouts.
C. C. Torrance (Cleveland, Ohio).

Ballabh, Ram. On two-dimensional self-superposable fluid motions. Proc. Benares Math. Soc. (N.S.) **4**, 27–31 (1943). [MF 10345]

Author obtains self-superposable solutions of the hydrodynamical equations for the case $\xi = \eta = 0$, $\zeta \neq 0$ [see the preceding review]. The case $\zeta = \zeta(y)$ includes motions in which the displacement of each particle is similar to that of a pendulum bob executing damped oscillations. The case $\zeta = \zeta(x, y)$ includes certain types of interacting eddies.
C. C. Torrance (Cleveland, Ohio).

Green, J. R. and **Southwell, R. V.** Relaxation methods applied to engineering problems. IX. High-speed flow of compressible fluid through a two-dimensional nozzle. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 367–386 (1944). [MF 10582]

The subsonic flow through a two-dimensional nozzle is to be determined by solving the differential equation

$$(\chi^2 \psi_x)_x + (\chi^2 \psi_y)_y = 0$$

for the stream function ψ, the quantity χ^2 being a given function of $\psi_x^2 + \psi_y^2$. The condition $\psi = 0$ is imposed on the axis $y = 0$, and $\psi = M = $ const. on the nozzle wall. Appropriate conditions are imposed at entrance and exit. The nozzle area is conformally mapped into a rectangle in an (α, β)-plane and the differential equations are replaced by finite difference equations with respect to a rectangular net in that plane. The finite difference equations are solved numerically by a relaxation process for various particular numerical data. The streamlines of the resulting flow are drawn; regions of supersonic flow are found to occur in the neighborhood of the wall at the throat. Difficulties are encountered when the same approach is used for supersonic flow.

An alternative method is introduced. The differential equation is written in the form

$$(\chi^2 \psi_\beta)_\beta = -(\chi^2 \psi_\alpha)_\alpha$$

and iterations are applied, assuming that the right hand side may be neglected in first approximation. The results of the first step agree largely with those of the method described above. It is proposed to employ this alternative method for supersonic flow. *K. O. Friedrichs.*

Kochin, N. E. The influence of the period of a lattice on its hydrodynamic characteristics. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 165–192 (1941). (Russian. English summary) [MF 7706]

The author gives for the complex velocity $v(z)$ of the flow around a lattice the formula

$$v(z) = \text{const.} + (1/2i\tau) \int_{C_1} v(\zeta) \coth (\pi(z - \zeta)/\tau) d\zeta,$$

where $i\tau$ denotes the period of the lattice. Here C_1 is a closed curve around one of the congruent profiles. The main object of the paper is the reduction of the general case to the classical case of the flow around one single profile.
A. Weinstein (Toronto, Ont.).

Loitsianskii, L. G. Integral methods in the theory of the boundary layer. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1070, 28 pp. (1944). [MF 10770]

English translation of a paper in J. Appl. Math. Mech. [Akad. Nauk SSSR Zhurnal Prikl. Mat. Mech.] **5**, 453–470 (1941); these Rev. **4**, 120. The original paper transliterated the author's name Loytzansky.

van Wijngaarden, A. Laminar flow in radial direction along a plane surface. Nederl. Akad. Wetensch., Proc. **45**, 269–275 (1942). [MF 10395]

A distribution of source and sink is located on the Z-axis. The laminar viscous incompressible flow then radiates from

the Z-axis with the XY plane as the plane boundary. If r is the distance from the axis, $\eta = z/r\sqrt{\epsilon}$, where ϵ is the reciprocal of the Reynolds number of the flow, then the velocity components u and w in the radial and axial directions are equal to $u_0 f'(\eta)/\rho$ and $u_0\sqrt{\epsilon}(\eta f' - f)/\rho$; r_0 and u_0 are the convenient measures of length and velocity, and $\rho = r/r_0$. For very large Reynolds numbers f satisfies the differential equation

$$(1) \qquad f''' + f''f + f'^2 - 1 = 0$$

with the boundary conditions (2) $f = 0$, $f' = 0$ for $\eta = 0$ and (3) $f = \eta - C$ for $\eta \to \infty$. Here (3) is the result of the requirement that $w/u = C$ for $\eta \to \infty$; (1) is a special case of Hartree's equation [Proc. Cambridge Philos. Soc. **33**, 223–239 (1937)]. The conditions (2) and (3) are sufficient to determine a unique solution, provided C is not equal to one of the critical values. The first critical value of C is approximately 1.09. For values of C between the first and second critical values, the solution has a pole at a finite value of η. For C between the second and third critical values, the solution has two poles, etc. However, the physical meaning of the solutions with poles is not certain as the approximations introduced to obtain (1) are no longer valid. Numerical values for the case of $C = 0$ are given in tabular form. For $-1.0 \leq C \leq 1.08$, the solutions are given in a graph.

In a final remark, the author investigates the more general case of expressing the stream function as $\rho^\alpha f(\eta)$, where $\eta = (z/r_0\sqrt{\epsilon})\rho^{-\gamma}$. There is the condition $\alpha + \gamma = 2$ to maintain the similarity law so defined. The case investigated by the author is the case $\alpha = 1$. Homann's case [Z. Angew. Math. Mech. **16**, 153–164 (1936)] corresponds to $\alpha = 2$. In order that the usual simplifications which lead to Hartree's equation be valid even for $\eta \to \infty$, α must be one of the values 1, 2, $-\frac{1}{2}$.   H. S. Tsien (Pasadena, Calif.).

Mohr, Ernst. Laminare und turbulente Strömungen mit besonderer Berücksichtigung des Einflusses von festen Wänden. Z. Phys. **121**, 301–350 (1943). [MF 9798]

Experimental evidence has shown, according to the author, that laminar flow satisfies a nonslip condition on solid boundaries whereas turbulent flow does not. The main part of the paper is consequently devoted to explain and reduce this discrepancy. A discussion of laminar and turbulent friction is presented first, in which the author expresses the opinion that the laminar stress tensor is not necessarily symmetric. [This opinion has already led to an extensive controversy; see E. Fues, Z. Phys. **118**, 409–415 (1941) and E. Mohr, Z. Phys. **119**, 575–580 (1942).] Furthermore, the analogy between fluid flow and heat transfer is discussed. The main part of the paper is then devoted to show that the nonslip condition is untenable. The author starts out from the well-known experiments of Kundt and Warburg on laminar flow of rarefied gases through tubes and concludes that the assumption of a finite wall velocity which was introduced by Kundt and Warburg to explain deviations from Poiseuille flow should be generally introduced into hydrodynamics.

The common assumption that the nonslip condition is the correct boundary condition within the frame of a phenomenological theory is, according to the author, wrong. The virtual experiments which the author uses to support his point of view have all in common that they lead to infinite shear on the wall. [Reviewer's remark. It appears that the difficulties encountered by the author in applying the nonslip condition are due to a misunderstanding of the scope and applicability of a continuum theory.]

The author then formulates new boundary conditions for viscous flow introducing a slip coefficient. Using these boundary conditions, an iteration method for the solution of the Stokes-Navier equation is sketched without arriving at any definite solution. He then proceeds to discuss critically Prandtl's boundary layer theory. The condition that the boundary layer approaches the free stream asymptotically so that the outer edge of the layer is often taken at $y = \infty$ seems to the author unsatisfactory because a boundary layer develops also initially when a body is moved in a closed vessel [!]. Similar objections are raised against Prandtl's treatment of laminar separation. Finally, a discussion of the boundary conditions for turbulent flow is given. Again an attempt is made to introduce a finite slip coefficient without arriving at any definite conclusions. [Reviewer's remark. The existence of a laminar sublayer is nowhere considered.]   H. W. Liepmann.

Eser, Franz. Zur Strömung kompressibler Flüssigkeiten um feste Körper mit Unterschallgeschwindigkeit. Luftfahrtforschung **20**, 220–230 (1943). [MF 9922]

This paper attempts to survey the known results of the problem of obtaining the subsonic potential flow of a perfect compressible fluid around a body in which the adiabatic pressure-density is assumed. The author deals briefly with the linearization method of Prandtl-Glauert, the successive approximation methods of Jenzen, Rayleigh, Poggi, the hodograph method of Molenbröck-Tchaplygin and discusses the methods specifically applied to the circular and elliptic cylinder and the symmetric Joukowski profiles. A fairly extensive bibliography is inserted at the end of the paper (64 entries).   A. Gelbart (Syracuse, N. Y.).

Dumitrescu, D. T. Strömung an einer Luftblase im senkrechten Rohr. Z. Angew. Math. Mech. **23**, 139–149 (1943). [MF 10181]

In this paper a study is made of the stationary potential flow of a liquid, in a vertical tube, about a long air bubble which slowly ascends the tube. The problem is to determine the speed of ascent and the shape of the bubble. The solutions of the potential equations for velocity potential and stream function are expressed in terms of Bessel functions. The coefficients in these expansions are expressed in terms of the coefficients of a power series expansion for the bubble contour near the stagnation point at the top of the bubble. The power series itself is determined by considering asymptotic properties of the flow far below the top of the (infinitely long) bubble. It turns out that near the top the bubble is hemispherical. The dependence of the speed of ascent on the fundamental constants of the problem is established by dimensional analysis. The precise value is then determined in terms of the coefficients in the Bessel expansion. A discussion of experimental work is also given. The agreement with theory is good.   W. Kaplan.

Evangelisti, Giuseppe. Sulle propagazione delle piccole onde nei canali a sezione variabile. Ann. Mat. Pura Appl. (4) **21**, 25–37 (1942). [MF 10506]

The author considers the form of tidal waves in a canal whose breadth b and depth h depend on the distance x along the canal [see Lamb, Hydrodynamics, Cambridge, 1932, ch. 8]. It is assumed that b and h are proportional to two different powers of x respectively. The partial differential equation for the wave amplitude is then integrated by separation of variables. The solution is expressed in terms

of Hankel functions, and it is shown that for large x the wave profile moves with negligible distortion.

$W. Kaplan$ (Providence, R. I.).

Loring, Samuel J. Use of generalized coordinates in flutter analysis. S.A.E. J. **52**, 113–132 (1944). [MF 10172]

This paper presents some further developments of the method of generalized coordinates in flutter analysis, a method which has been given by the author in a previous paper [S.A.E.J. **49**, 345–356 (1941); these Rev. **3**, 222]. The paper is composed of three parts. Part I shows in detail the calculations by which this method was applied to a flutter model. The check with the experimental results is also shown. Part II is a manual for solving more complicated flutter problems by this method; this manual has been in successful use for some time. Part III is a discussion of the most important and difficult part of the generalized coordinate method, namely, the choice of the generalized coordinates. $E. Reissner$ (Cambridge, Mass.).

Ketchum, P. W. On the discontinuous flow around an airfoil with flap. Quart. Appl. Math. **1**, 149–167 (1943). [MF 8805]

The author considers the discontinuous flow around a broken line representing the main wing and the flap of an airfoil. One of the points of detachment of the wake is the tail end of the lower side; the other is located on the upper side of the flap and is to be determined by the condition that the width of the wake tends to zero at infinity. The velocity becomes necessarily infinite at the leading edge and at the hinge of the flap. The determination of the four parameters in the corresponding formulae of Levi-Civita is the most difficult part of the procedure. Numerical and graphical discussions are given for certain small values of the angles of attack. The drag and lift are computed using the same convention as in the case of thin airfoils in the Kutta-Joukowsky theory, and it is found that the drag vanishes. The author believes that his formulae represent an approximation to the flow (with finite velocities) around an airfoil of finite but small thickness and refers extensively to some papers of C. Schmieden [Ing.-Arch. **3**, 356–370 (1932); **5**, 373–375 (1934)], stating that Schmieden has made a definite advance in the general theory of discontinuous motion. The opposite opinion on Schmieden's claims has been expressed by the reviewer [Enseignement Math. **36**, 107 (1936)] and by J. Leray [Comment. Math. Helv. **8**, 149–180 (1935)]. $A. Weinstein.$

Schwarz, L. Zur Theorie der Beugung einer ebenen Schallwelle an der Kugel. Akustische Z. **8**, 91–117 (1943). [MF 9921]

The paper gives many numerical tables and charts connected with the diffraction of a plane wave by a sphere. The theory is essentially classical. $D. G. Bourgin.$

Buchholz, Herbert. Die Ausbreitung der Schallwellen in einem Horn von der Gestalt eines Rotationsparaboloides bei Anregung durch eine im Brennpunkt befindliche punktförmige Schallquelle. Ann. Physik (5) **42**, 423–460 (1942). [MF 10221]

This is a study of a long acoustically rigid horn in the shape of a paraboloid of revolution with an harmonic source at the focus. Roughly speaking, the key to the practical calculation lies in expressing the simple source function e^{ikR}/R as a Mellin integral with integrands Whittaker functions. The solution is then built up by superposition. Two sets of expansions are considered, the one for the neighborhood of the vertex of the paraboloid and the other for the region near the mouth of the horn. [The qualitative mathematical features could have been predicted from the physical picture, for evidently near the mouth of the horn the phenomenon is essentially that of a diverging set of waves stemming from the source, and the field can be described in terms of superposition of characteristic solutions, the mathematical distinction between original and reflected waves being unnecessary. Behind the focus the situation is more complicated. The sound field cannot be naturally described by characteristic solutions because it is the combination of reflected and incident waves that satisfies the boundary conditions and not either separately.] An interesting by-product is the result that, in contrast with the situation for conical horns, there is transverse velocity (except at the boundary, of course) for the zeroth order waves. $D. G. Bourgin$ (Urbana, Ill.).

Stenzel, H. Das Schallfeld eines Strahlers in einer Mediumschicht mit schallweicher und schallharter Begrenzung. Ann. Physik (5) **43**, 1–31 (1943). [MF 9928]

The writer treats the problem of a sound field between two parallel plates. The calculations are made by specializing the analysis of R. Weirich [Ann. Physik (4) **85**, 552–580 (1928)], who considered an analogous problem for electromagnetic waves. Numerical tables and graphs illustrate the results. $D. G. Bourgin$ (Urbana, Ill.).

Theory of Elasticity

Platrier, Charles. Tension différentielle dans les milieux parfaitement élastiques en équilibre sans forces de masse. C. R. Acad. Sci. Paris **216**, 109–111 (1943). [MF 10004]

When the body forces are omitted in an elastic medium, the components τ_{ij} of the symmetric stress tensor τ satisfy the differential equations $\tau_{ij,j}=0$. It is well known that a constant value of the form $2F=\tau_{ij}x_i x_j$ gives to some scale a geometrical representation of this stress tensor τ. In this paper the author attempts the geometrical representation of the differential changes of the stresses by means of two forms: (1) $3C=\Delta_{ijk}x_i x_j x_k$, where $3\Delta_{ijk}=\tau_{ij,k}+\tau_{jk,i}+\tau_{ki,j}$; (2) $2Q=\theta_{ij}x_i x_j$ ($\theta_{11}+\theta_{22}+\theta_{33}=0$). The ten coefficients Δ_{ijk} in (1) are the ten independent components of the third order tensor Δ, and the six coefficients θ_{ij} are the components of the symmetrized second order tensor curl τ. The author states that these 15 independent quantities are fully equivalent to the 18 elements $\tau_{ij,k}$ which are subject to the three conditions $\tau_{ij,j}=0$, and that the forms (1) and (2) define the differential tractions and differential torsions of an elastic fibre whose directions are proportional to x_i.

$D. L. Holl$ (Ames, Iowa).

Barjansky, A. The distortion of the Boussinesq field due to a circular hole. Quart. Appl. Math. **2**, 16–30 (1944). [MF 10333]

The author calculates the distortion introduced into the plane Boussinesq field [Timoshenko, Theory of Elasticity, McGraw-Hill, New York, 1934, p. 82] by the presence of a circular hole. The stress function is decomposed into two parts: (a) the stress function ϕ for the unperturbed Boussinesq field and (b) a second stress function χ such that $\Phi=\phi+\chi$ satisfies the plane biharmonic equation and certain boundary conditions. Use is made of Jeffrey's discussion of

the solution of the biharmonic equation in bipolar coordinates [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 265–293 (1920)].   *A. E. Heins* (Cambridge, Mass.).

Schulz, K. J. On the state of stress in perforated strips and plates. Nederl. Akad. Wetensch., Proc. **45**, 233–239 (1942). [MF 10390]

Schulz, K. J. On the state of stress in perforated strips and plates. II. Nederl. Akad. Wetensch., Proc. **45**, 341–346 (1942). [MF 10399]

Schulz, K. J. On the state of stress in perforated strips and plates. III. Nederl. Akad. Wetensch., Proc. **45**, 457–464 (1942). [MF 10410]

Schulz, K. J. On the state of stress in perforated strips and plates. IV. Nederl. Akad. Wetensch., Proc. **45**, 524–532 (1942). [MF 10416]

The author states that in a former paper he has treated the problem of the stress distribution in infinite plates containing circular holes of arbitrary position and arbitrary radii and, in particular, problems in which equal holes are arranged in one or more infinite rows of constant pitch and in which the states of stress and strain are periodic with a period equal to that of the pitch. Frequent reference is made to this paper, but its place of publication is not given. In the present paper, the method (superposition of suitable solutions of the biharmonic equation) is extended to the problem of the infinite strip in tension with a row of holes parallel to its edges. The case of an infinite plate containing a single circular hole with arbitrarily prescribed stress on its boundary is first treated. Reference is not made to W. G. Bickley's treatment of this problem [Philos. Trans. Roy. Soc. London. Ser. A. **227**, 383–415 (1928)]. The result is utilized in obtaining the stress function for the problem of an infinite plate with a row of holes all with the same distribution of stress on their boundaries. Next, the stress function is set up for a semi-infinite plate with a periodic distribution of stress on its edge. By combining the results thus obtained in such a way as to satisfy the requirement that the edges of the holes and of the infinite strip are free from the action of external forces, the solution is found of the problem of a perforated infinite strip in pure tension. According to an introductory statement, the problems of the infinite perforated strip in pure bending and of the semi-infinite plate with a row of holes parallel to its edge and loaded by equal resultant forces are to be treated in subsequent sections of this paper.   *H. W. March.*

Greenspan, Martin. Effect of a small hole on the stresses in a uniformly loaded plate. Quart. Appl. Math. **2**, 60–71 (1944). [MF 10337]

This paper deals with the solution of a generalized plane stress problem of an infinite plate having an interior curvilinear notch or ovaloid. The plate is uniformly loaded in the sense that at infinity the stresses τ_{xx}, τ_{yy} and τ_{xy} have the respective constant values S_x, S_y and T_{xy}. The stress function ϕ satisfies

$$\nabla^4\phi = \nabla^2(h^2\nabla^2\phi(\alpha, \beta)) = 0,$$

where α, β are curvilinear coordinates introduced by

$$z = x+iy = f(\alpha+i\beta) = f(w) = e^w + k_1 e^{-w} + k_2 e^{-3w}$$

and $h = |dw/dz|$. The stresses $\tau_{\alpha\alpha}$, $\tau_{\beta\beta}$ and $\tau_{\alpha\beta}$ are determined in terms of ϕ and h. By superposing linear combinations of appropriate particular solutions of ϕ, the conditions at the infinite boundary and at the edge of the notch are satisfied. The tangential stresses are evaluated along the boundary

of certain special notches and comparisons made with the known solutions for elliptic and circular notches.
     D. L. Holl (Ames, Iowa).

Chien, Wei-Zang. The intrinsic theory of thin shells and plates. II. Application to thin plates. Quart. Appl. Math. **2**, 43–59 (1944). [MF 10336]

In part I [Quart. Appl. Math. **1**, 297–327 (1943); these Rev. **5**, 195] there were deduced six fundamental equations for the six unknowns $p_{\alpha\beta}$, $q_{\alpha\beta}$ ($\alpha, \beta=1, 2$) which represent, respectively, the extension and change of curvature of the middle surface of a shell. Once these six quantities have been determined, the stress and strain in the shell can be found.

In the present paper, the author applies the results of part I to thin plates, that is, the middle surface in the unstrained state is a plane and $\epsilon \ll 1$, where ϵ is the ratio of the average thickness to the smallest lateral dimension of the shell. Solutions of the fundamental equations are sought in the form of power series in ϵ:

$$p_{\alpha\beta} = \sum_{s=\gamma}^{\infty} p_{(s)\alpha\beta}\epsilon^s, \quad q_{\alpha\beta} = \sum_{s=\epsilon}^{\infty} q_{(s)\alpha\beta}\epsilon^s.$$

It is found that the six fundamental equations take on twelve distinct forms (types P1–P12), depending on the values assigned to p and q. Types P1–P3 are concerned with finite deflections, types P4–P8 with small deflections, types P9–P11 with very small deflections and type P12 with zero deflections. Types P1–P3 are new and are discussed in detail.

It is noted that the theory of generalized plane stress, the Lagrange-Kirchhoff theory of small deflection and the von Kármán theory of "large" deflection can be derived, respectively, from types P12, P11, P5.   *G. E. Hay.*

Byrne, Ralph, Jr. Theory of small deformations of a thin elastic shell. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 103–152 (1944). [MF 10458]

In this paper the theory of small deformations of thin elastic shells is developed anew. The work is based on the generally accepted assumption that lines which are normal to the middle surface before deformation remain normal to the deformed middle surface after deformation and suffer no extension. Retaining all first order terms, an expression for the potential energy of the deformation is obtained from which, by means of the principle of virtual work, the differential equations and boundary conditions for a general shell are deduced. It is proved that the solution of the differential equations subject to prescribed boundary conditions of the type deduced is essentially unique.
     E. Reissner (Cambridge, Mass.).

Reutter, F. Der starre Kreiszylinder im isotropen elastischen Medium. Z. Angew. Math. Mech. **23**, 156–169 (1943). [MF 10179]

The following problem is considered. A rigid circular cylinder of radius c is inserted in the cut-out portion $0 \leq z \leq k$, $r \leq c$ of the elastic half space $z > 0$. It is assumed that the displacements normal to the boundary of the cylinder and the half space are continuous, the displacement of the cylinder in the direction of its own axis being prescribed, and that along the walls of the cylindrical stamp $\tau = $const. σ. The solution of the differential equations of the problem is taken in the form of Fourier-Bessel integrals. The boundary conditions become two linear integral equations of the first kind. They are solved by reducing them to a system of

infinitely many linear equations for an infinite number of unknowns. A few numerical results are included.

E. Reissner (Cambridge, Mass.).

Haringx, J. A. On the buckling and the lateral rigidity of helical compression springs. I, II. Nederl. Akad. Wetensch., Proc. **45**, 533–539, 650–654 (1942). [MF 10417]

Brown, C. L. The treatment of discontinuities in beam deflection problems. Quart. Appl. Math. **1**, 349–351 (1944). [MF 9913]

It is indicated that for beams with but sectionally continuous load distribution the sectionalizing treatment can be avoided by the use of Heaviside's unit step function.

E. Reissner (Cambridge, Mass.).

Binnie, A. M. Stresses in the diaphragms of diaphragm-pumps. Quart. Appl. Math. **2**, 37–42 (1944). [MF 10335]

Garavito Armero, Julio. Oscillations of a prismatic bar on a circular cylinder. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **5**, 370–373 (1943). (Spanish) [MF 10595]

Conn, J. F. C. Vibration of a truncated wedge. Aircraft Engrg. **16**, 103–105 (1944). [MF 10581]

This paper deals with vibrations of a truncated wedge, free at the thin end and encastré at the thick end. Natural frequencies and the corresponding characteristic functions are calculated for longitudinal, torsional and two types of flexural vibration. It is stated that the results are of some value in estimating the vibration frequencies of screw propeller blades. The treatment of the torsional vibrations is admittedly only an approximation; and the reviewer has been unable to follow that part of the work. The reviewer believes that the work on the flexural vibrations is definitely erroneous, because of the use of incorrect boundary conditions.

L. A. MacColl (New York, N. Y.).

Myklestad, N. O. A new method of calculating natural modes of uncoupled bending vibration of airplane wings and other types of beams. J. Aeronaut. Sci. **11**, 153–162 (1944). [MF 10247]

The author presents a computational method for finding the natural modes (both frequencies and deflection curves) of beams. The method is somewhat analogous to Holzer's method of finding the natural modes of torsional vibrations. It is assumed that the elastic axis of the unbent beam is a straight line which bends into a plane curve during vibration. In the case of the vibration of airplane wings, the wings are considered to be weightless and to carry a number of concentrated masses equally spaced from the base to the tip. The two wings are assumed to vibrate symmetrically with frequency $\omega/2\pi$ cycles per second under the action of a vertical shaking force $2F \cos \omega t$ acting through the middle of the fuselage. The total inertia force of all the masses is then computed for various values of ω and is plotted against ω. In this graph, the ω-intercepts are the natural frequencies of the system, since they correspond to zero total inertia force, that is, to vibrations which can be maintained without any external force. Antisymmetric bending vibrations are treated similarly by applying to the fuselage a shaking moment $2M \cos \omega t$, computing the total inertia moment of all the masses and plotting it against ω. The case is also considered when the engines have flexible mountings permitting vertical oscillations.

The method is applied in similar form to cantilever beams, simply supported beams with and without overhang and to beams on several supports.

G. E. Hay.

Ryz, P. M. Oscillation of bars of non-symmetrical shape due to bending and twisting. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 231–232 (1943). (Russian. English summary) [MF 10078]

The author attempts to explain the appearance of resonance at a frequency equal to half of the natural frequency of the oscillating system. This phenomenon can be frequently observed in oscillations of a propeller blade having a very unsymmetrical shape.

Author's summary.

Riz, P. M. General solution of the torsion problem in the nonlinear theory of elasticity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 149–154 (1943). (Russian. English summary) [MF 10072]

The problem of torsion of a prismatic bar is considered by taking into account the squares of derivatives of displacements. The method of solution was discussed by the present author and N. B. Zvolinsky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] (N.S.) **2**, 417–426 (1939)]. The special cases of circular and elliptic cylinders were studied by P. M. Riz and D. Y. Panov [Trudy Central Aero-Hydrodynamical Institute **408** (1939), **459** (1939)]. The nonlinear theory brings to light the following phenomena which do not appear in the classical theory of Saint Venant: (1) the axial compression of the bar; (2) the additional deformation distorting the cross-section; (3) the bending of a bar, which depends on the choice of the axes of torsion and which can be made to vanish by a suitable choice of axes.

I. S. Sokolnikoff (Madison, Wis.).

von Kármán, Th. and **Christensen, N. B.** Methods of analysis for torsion with variable twist. J. Aeronaut. Sci. **11**, 110–124 (1944). [MF 10246]

The paper is concerned with the stresses in prismatic bars of thin-walled cross section which are loaded in torsion. When the angle of twist per unit length is constant along the bar, all cross sections warp in the same manner. If one or both end sections are restrained from warping, the angle of twist per unit length will vary along the bar. In this case of variable twist the shear flow in the thin-walled sections is regarded as consisting of two parts: the primary shear flow produced by pure twist without axial constraint, and the secondary shear flow produced by the axial stresses due to the "bending" in the planes of the thin-walled elements. The well-known method of calculating the primary shear stresses is reviewed briefly; by far the greater part of the paper, however, is concerned with the determination of the secondary shear stresses. The torque T is written as the sum of the torques T and T' corresponding to the primary and secondary stresses, respectively. Let $\theta(x)$ denotes the variable rate of twist, $T = CG\theta$ and $T' = -C'Ed^2\theta/dx^2$, where the values of the constants C and C' depend only on the shape of the cross section, and G and E denote the modulus of rigidity and Young's modulus, respectively. Thus $T = T + T' = CG\theta - C'Ed^2\theta/dx^2$. When the total torque T is given, this differential equation, together with the boundary conditions, determines θ as a function of x. From $\theta(x)$ an expression is obtained for the normal strains in the direction of the axis of the bar. Hooke's law furnishes then the normal stresses, and the equilibrium condition for the direction of the axis of the bar gives the secondary shear stresses. This part of the

analysis can be reduced to a sequence of graphical integrations analogous to those used in beam theory.
W. Prager (Providence, R. I.).

Weigand, A.   Das Torsionsproblem für Stäbe von kreisabschnittförmigem Querschnitt. Luftfahrtforschung 20, 333–340 (1944).   [MF 10578]
The author compares the results of the exact solution of the torsion problem for a right prism of half circular section with an approximate solution obtained by the method of least squares. A linear combination of particular solutions of the differential equation is employed as an approximate torsion function and the parameters are determined so that the contour integral of the square of the boundary error is a minimum. A similar treatment is made for a prism whose cross section is a segment of a circle and the results of the approximate solution are compared with those obtained from tests on laboratory specimens.   *D. L. Holl.*

Bartels, R. C. F.   Torsion of hollow cylinders. Trans. Amer. Math. Soc. 53, 1–13 (1943).   [MF 7821]
The torsion problem for a cylinder leads to a Dirichlet problem for its cross-section. The author gives a rather complicated method for the solution of the Dirichlet problem in an annulus. He solves the torsion problem for an eccentric ring and a hollow lune, indicating the functions which map the above domains into an annulus. The bibliography, although extensive, is not complete. For instance, the author does not mention the papers by Kuffaref [Appl. Math. Mech. (N.S.) 1, 43–76 (1937)] and Golusin [Works Sci. Research Inst. Math. Mech. Leningrad 1937, 90–97].
S. Bergman (Providence, R. I.).

Ishlinsky, A. J.   The stressed state of a cylinder at large angles of torsion. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 223–225 (1943). (Russian. English summary)   [MF 10077]
The paper deals with the twisting of a circular cylinder and with so-called secondary effects such as the shortening of the length of the cylinder, the decrease in diameter and the compressive radial stresses due to torsion.
Author's summary.

Ishlinsky, A. J.   On the stability of plastico-viscous flows of a rectangular strip and a round bar. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 109–130 (1943). (Russian. English summary)   [MF 9731]
In the first part the author investigates the stability of a uniform two-dimensional visco-plastic flow of a rectangular strip, using the usual method of small disturbances. He finds that during extension the flow passes through stable and unstable states and finally becomes unstable. A. A. Ilyushin [Sci. Ann. Moscow State Univ. 39 (1940)] treated the same problem in terms of Lagrangian variables, whereas the present author uses the Euler representation.
In the second part the author sets up the differential equations of a three-dimensional visco-plastic flow and obtains a simple solution of these equations which corresponds to the uniform deformation of a cylindrical bar of circular cross section. The stability of this solution is investigated.
L. Bers (Providence, R. I.).

Handelman, G. H.   A variational principle for a state of combined plastic stress. Quart. Appl. Math. 1, 351–353 (1944).   [MF 9914]
The author shows that the following form of Sadowsky's principle [J. Appl. Mech. 10, A-65–A-68 (1943); these Rev.

4, 263] leads to the correct differential equation for a perfectly plastic, incompressible beam under combined torsion and bending by couples: "among all statically possible stress distributions in a beam under a given torque (satisfying the equations of equilibrium, the condition of plasticity and the boundary conditions), the actual stress distribution when plastic flow occurs is the one for which the bending moment is stationary."
Introducing expressions for the strain velocity components similar to those satisfied by the strain vector components in the corresponding elastic case, the author determines first the stresses from the stress-rate of strain relations for a perfectly plastic material. The equilibrium relations are satisfied by introducing an Airy stress function. It is found that the stresses will satisfy the yield condition if the Airy stress function satisfies a nonlinear second order partial differential equation. Finally, by determining the Euler equation for a stationary value of the bending moment under a given torque, it is shown that the same nonlinear equation is satisfied by the Airy stress function. [The last term in the right hand side of equation (7) should read $\lambda\mu$.]
N. Coburn (Austin, Tex.).

Gómez Sánchez, José Domingo.   Parametric representation of elastic waves in anisotropic media of the cubic crystalline systems. Revista Ci., Lima 45, 309–331 (1943).   (1 plate)   (Spanish)   [MF 10080]

Scholte, J. G.   On the Stoneley-wave equation. I. Nederl. Akad. Wetensch., Proc. 45, 20–25 (1942).   [MF 10373]
Scholte, J. G.   On the Stoneley-wave equation. II. Nederl. Akad. Wetensch., Proc. 45, 159–164 (1942).   [MF 10382]
The Stoneley wave-system [Proc. Roy. Soc. London. Ser. A. 106, 416–428 (1924)] is derived by a new method generalizing the calculations of Knott [Philos. Mag. (5) 48, 64–97, 567–568 (1899)]. The corresponding velocity equation of the Stoneley wave is not always solvable as has been shown by Stoneley. Stoneley erroneously concluded that the small range of parameter values for which it is solvable does not occur in practice. The author shows that Stoneley waves are possible for widely different values of material constants and determines completely the range for which the Stoneley equation can be solved.   *E. Kogbetliants.*

Scholte, J. G.   On surface waves in a stratified medium. I. Nederl. Akad. Wetensch., Proc. 45, 380–386 (1942). [MF 10407]
Scholte, J. G.   On surface waves in a stratified medium. II. Nederl. Akad. Wetensch., Proc. 45, 449–456 (1942). [MF 10409]
Scholte, J. G.   On surface waves in a stratified medium. III. Nederl. Akad. Wetensch., Proc. 45, 516–523 (1942).   [MF 10415]
The author determines all damped wave systems possible in a stratified medium. As particular cases the well-known classical Love, Rayleigh and Stoneley wave systems reappear. Generalized Rayleigh and Stoneley waves, possible only for certain ranges of values of the material constants, are investigated. The corresponding variation ranges for these parameters are determined by analyzing the period equation corresponding to the general case. Numerical computations are carried out for incompressible media. At the end the dispersion curves are discussed and their general shape is ascertained.   *E. Kogbetliants.*

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